Finiteness spaces and generalized power series

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Overview

- Ribenboim constructed rings of generalized power series for studies in number theory.
- While his construction gives a rich class of rings, it also seems ad hoc and non-functorial.
- We show that the conditions he imposes in fact can be used to construct internal monoids in a category of Ehrhard's *finiteness spaces* and the process is functorial.
- Furthermore any internal monoid of finiteness spaces induces a ring by Ehrhard's *linearization* process. So we get lots of new examples of generalized power series.

Power series rings

We have the usual power series multiplication making $K[[z]]$ a ring:

$$(\sum_{n=0}^{\infty} a_n z^n)(\sum_{n=0}^{\infty} b_n z^n) = \sum_{n=0}^{\infty} c_n z^n \quad \text{with} \quad c_n = \sum_{j+k=n} a_j b_k$$

But suppose we wish to add negative exponents:

$$(\sum_{-\infty}^{\infty} a_n z^n)(\sum_{-\infty}^{\infty} b_n z^n) \text{ can lead to infinite coefficients.}$$

A solution is *Laurent Series* which bound the indexing set below:

$$\sum_{n=k}^{\infty} a_n z^n \text{ where k can be negative (but is finite)}$$

This ensures that, for all n, the set of pairs (j, k) with $j + k = n$ is finite and hence the above product is well-defined.

We'll need the following technical condition:
Let $(M, +, \leq)$ be a partially ordered monoid. M is *strictly ordered* if

$$s < s' \Rightarrow s + t < s' + t \; \forall s, s', t \in M$$

and similarly adding t on the left. We will henceforth assume that all the monoids we work with are strictly ordered.

Definition

An ordered monoid is *artinian* if all strictly descending chains are finite; that is, if any list $(m_1 > m_2 > \cdots)$ must be finite. It is *narrow* if all discrete subsets are finite; that is, if all subsets of elements mutually unrelated by $\leq$ must be finite.

Ribenboim's generalized power series II

Definition

Let A be an abelian group, and recall that the *support* of a function $f : M \to V$ is defined by $supp(f) = \{m \in M | f(m) \neq 0\}$. Define the *space of Ribenboim power series from M with coefficients in A*, $G(M, A)$ to be the set of functions $f : M \to V$ whose support is artinian and narrow.

If A is also a $\mathbb{K}$-algebra, then $G(M, A)$ is a $\mathbb{K}$-algebra with the following convolution product:

$$(f \cdot g)(m) = \sum_{(u,v) \in X_m(f,g)} f(u) \cdot g(v)$$

where

$$X_m(f, g) := \{(u, v) \in M \times M | u + v = m \text{ and } f(u) \neq 0, g(v) \neq 0\}$$

Ribenboim's generalized power series III

This requires the following observation. It is where the restrictions imposed are used:

Proposition

The set $X_m(f,g)$ is finite for $f, g \in G(M,A)$. So when A is a ring, $G(M,A)$ is a ring with the above formula as multiplication.

There are lots of examples.

- Let $M = \mathbb{N}$. The result is the usual ring of power series with coefficients in A.
- Let $M = \mathbb{Z}$. The result is the ring of Laurent series with coefficients in A.

Ribenboim's generalized power series IV: More examples

- Let $M = \mathbb{N}^n$, with pointwise order. The result is the usual ring of power series in n-variables with coefficients in A.

  This example is due to Ribenboim and was his motivation:

- Let $M = \mathbb{N} \setminus \{0\}$ with the operation of multiplication, equipped with the usual ordering. Then $G(M, \mathbb{R})$ is the ring of arithmetic functions (i.e. functions from the positive integers to the complex numbers), and multiplication is Dirichlet's convolution:

$$(f \star g)(n) = \sum_{d|n} f(d) g(\frac{n}{d})$$

Ehrhard's finiteness spaces I

- Let X be a set and let $\mathcal{U}$ be a set of subsets of X, i.e., $\mathcal{U} \subseteq \mathcal{P}(X)$. Define $\mathcal{U}^{\perp}$ by:

$$\mathcal{U}^{\perp} = \{u' \subseteq X \mid \text{the set } u' \cap u \text{ is finite for all } u \in \mathcal{U}\}$$

Lemma

- $\mathcal{U} \subseteq \mathcal{U}^{\perp\perp}$
- $\mathcal{U} \subseteq \mathcal{V} \Rightarrow \mathcal{V}^{\perp} \subseteq \mathcal{U}^{\perp}$
- $\mathcal{U}^{\perp\perp\perp} = \mathcal{U}^{\perp}$

- A *finiteness space* is a pair $\mathbb{X} = (X, \mathcal{U})$ with X a set and $\mathcal{U} \subseteq \mathcal{P}(X)$ such that $\mathcal{U}^{\perp\perp} = \mathcal{U}$. We will sometimes denote X by $|\mathbb{X}|$ and $\mathcal{U}$ by $\mathcal{F}(\mathbb{X})$. The elements of $\mathcal{U}$ are called *finitary* subsets.

Ehrhard's finiteness spaces II: Morphisms

- A *morphism* of finiteness spaces $R\colon \mathbb{X} \to \mathbb{Y}$ is a relation $R\colon |\mathbb{X}| \to |\mathbb{Y}|$ such that the following two conditions hold:
  (1) For all $u \in \mathcal{F}(\mathbb{X})$, we have $uR \in \mathcal{F}(\mathbb{Y})$, where $uR = \{y \in |\mathbb{Y}| \mid \exists x \in u, xRy\}$.
  (2) For all $v' \in \mathcal{F}(\mathbb{Y})^{\perp}$, we have $Rv' \in \mathcal{F}(\mathbb{X})^{\perp}$.

It is straightforward to verify that this is a category. We denote it FinRel.

Lemma

In the definition of morphism of finiteness spaces, condition (2) can be replaced with:

$$(2')\ \textit{For all } b \in |\mathbb{Y}|, \textit{ we have } R\{b\} \in \mathcal{F}(\mathbb{X})^{\perp}.$$

Ehrhard's finiteness spaces III: It's a model of linear logic

Theorem

FinRel *is a* $*$*-autonomous category. The tensor*

$$\mathbb{X} \otimes \mathbb{Y} = (|\mathbb{X} \otimes \mathbb{Y}|, \mathcal{F}(\mathbb{X} \otimes \mathbb{Y}))$$

is given by setting $|\mathbb{X} \otimes \mathbb{Y}| = |\mathbb{X}| \times |\mathbb{Y}|$ *and*

$$\begin{aligned} \mathcal{F}(\mathbb{X} \otimes \mathbb{Y}) &= \{u \times v \mid u \in \mathcal{F}(\mathbb{X}), v \in \mathcal{F}(\mathbb{Y})\}^{\perp\perp} \\ &= \{w \mid \exists u \in \mathcal{F}(\mathbb{X}), \exists v \in \mathcal{F}(\mathbb{Y}), w \subseteq u \times v\}. \end{aligned}$$

We note that it also has sufficient structure to model the rest of the connectives of linear logic.

Ehrhard's finiteness spaces IV: Another choice of morphism

Ehrhard was motivated by linear logic to construct a $*$-autonomous category and hence chose relations as morphisms. But the choice has issues. Much like the usual category of relations, FinRel is lacking most limits and colimits. Another choice is possible:

Definition

We define the category FinPf*. Objects are finiteness spaces and a morphism* $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ *is a partial function satisfying the same conditions as above.*

Proposition

The category FinPf *is a symmetric monoidal closed, complete and cocomplete category.*

Posets as finiteness spaces I

Ribenboim's use of artinian and narrow subsets may seem unmotivated, but it in fact is precisely what we need to embed posets into finiteness spaces:

Theorem

Let $(P, \leq)$ be a poset. Let $\mathcal{U}$ be the set of artinian and narrow subsets. Then $(P, \mathcal{U})$ is a finiteness space.

Lemma

Under the above assumptions, $\mathcal{U}^{\perp}$ is the set of noetherian subsets of P.

Posets as finiteness spaces II: Functoriality

Unfortunately, if we consider the above construction from the usual category Pos of posets to any of the categories of finiteness spaces we have considered, it isn't functorial. Indeed, the inverse image under an order-preserving map of a noetherian subset may be not noetherian. However, the problem disappears if we consider *strict maps.*

Definition

If $(P, \leq)$ and $(Q, \leq)$ are two posets, a map $f: P \to Q$ is said to be *strict* if $p < p'$ implies $f(p) < f(p')$. In particular, it is a morphism of posets. We denote the category of posets and strict maps by StrPos.

Proposition

The above construction is a strict symmetric monoidal functor $E: \mathrm{StrPos} \to \mathrm{FinPf}$*.*

Posets as finiteness spaces III: Internal monoids

As such, it takes monoids to monoids:

Theorem

The functor E induces a functor $Mon(E)\colon Mon(\mathrm{StrPos}) \to Mon(\mathrm{FinPf})$ from the category of strict pomonoids to the category of partial finiteness monoids.

Definition

A *partial finiteness monoid* is an internal monoid in FinPf.

Linearizing finiteness spaces and generalizing the Ribenboim construction

Let A be an abelian group and $\mathbb{X} = (X, \mathcal{U})$ a finiteness space. Ehrhard defined the abelian group $A\langle\mathbb{X}\rangle$ as the set

$$A\langle\mathbb{X}\rangle = \{f : X \to A \,|\, supp(f) \in \mathcal{U}\}$$

together with pointwise addition.

Lemma

In the case of a poset $(P, \leq)$ with its finiteness structure as determined as above, we recover $G(P, A)$.

Linearizing II

Theorem

If $(\mathbb{M}, \mu\colon \mathbb{M} \otimes \mathbb{M} \to \mathbb{M}, \eta\colon I \to \mathbb{M})$ *is a partial finiteness monoid and* R *a ring (not necessarily commutative, but with unit), then* $R\langle\mathbb{M}\rangle$ *canonically has the structure of a ring.*

The multiplication in $R\langle\mathbb{M}\rangle$ *is given by*

$$(f \cdot g)(m) = \sum_{(m_1,m_2)\in X_m(f,g)} f(m_1) \cdot g(m_2).$$

where
$X_m(f,g) = \{(m_1, m_2) \in M^2 \mid m_1 + m_2 = m,\ f(m_1) \neq 0,\ g(m_2) \neq 0\}.$

Note the obvious similarity to Ribenboim's definition. But here it is the second condition in the definition of morphism of finiteness spaces that ensures the finiteness of the sum.

It's well-defined

Why is the set

$$X_m(f,g) = \{(m_1, m_2) \in M^2 \mid m_1 + m_2 = m,\ f(m_1) \neq 0,\ g(m_2) \neq 0\}$$

finite?

This set is exactly

$$\underbrace{(\mathrm{supp}(f) \times \mathrm{supp}(g))}_{\in \mathcal{W}} \cap \underbrace{\mu^{-1}(m)}_{\in \mathcal{W}^{\perp}}$$

Recall that μ is the multiplication. $\mathcal{W}$ is the finiteness space structure for $\mathbb{M} \otimes \mathbb{M}$.

Example I: Puiseux series (Newton)

A *Puiseux series* with coefficients in the ring R is a series (with indeterminate T) which allow for negative and fractional exponents of the form

$$\sum_{i \geqslant a}^{+\infty} r_i T^{i/n}$$

for some integer $a \in \mathbb{Z}$, some positive integer $n \in \mathbb{N} \setminus \{0\}$ and where $r_i \in R$. With the usual sum and product law, they form the ring of Puiseux series with coefficients in R.

Our postdoc Pierre-Alain Jacqmin showed that these rings fit into the finiteness space framework. Details in our paper on the archive.

Example II: Formal power series

Let A be a set (called in this case the *alphabet*). Then, let M be the free monoid generated by A. The finiteness space $(M, \mathcal{P}(M))$ has a monoid structure in FinPf given by the classical monoid structure of M. The only non-trivial part here is to check that the multiplication

$$\cdot : (M, \mathcal{P}(M)) \otimes (M, \mathcal{P}(M)) \to (M, \mathcal{P}(M))$$

is a morphism.

But since M is freely generated by A, for each $m \in M$, there are only finitely many $(m_1, m_2) \in M^2$ such that $m_1 \cdot m_2 = m$.

Then the ring $R\langle(M, \mathcal{P}(M))\rangle$ is called the *ring of formal power series* with exponents in M and coefficients in R.

Future work

- Étale groupoids yield C^*-algebras. The construction is very similar.
- Can generalized power series be differentiated?
- Do all rings that arise as above have a *Rota-Baxter operator*? One place RB-operators arise is in renormalization in quantum field theory. Rings of Laurent series have such an operator which is used in the Connes-Kreimer approach to renormalization. Guo and Liu studied when a projection operator on Ribenboim power series is in fact a Rota-Baxter operator. Do these necessarily exist for finiteness monoids and their rings?
- Morita theory.