Extension creation under base change[1]

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[1] joint with Branko Nikolić

Introduction

- By now there are numerous variants of category theory; for example,
  - categories enriched in a fixed monoidal category or bicategory,
  - categories parametrized over a fixed category,
  - stacks on a site,
  - categories with given extra properties or structure,
  - the derivators of homotopy theory,
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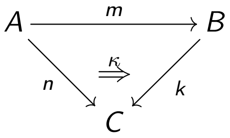
- The bicategorical concept of left extension is extremely expressive. This is well documented.
- Little has been done on preservation and reflection of left extensions by morphisms between bicategories.
- The goal is to explain how this can happen in comonadic situations.

Left extensions

Please remember this diagram for three more frames!!

A diagram



in a bicategory $\mathscr{N}$ exhibits k as a *left extension* of n along m when, for all $g\colon B \to C$, the function

$$\mathscr{N}(B,C)(k,g) \longrightarrow \mathscr{N}(A,C)(n, g\circ m)\ ,$$
$$(k \overset{\theta}{\Rightarrow} g) \mapsto (n \overset{\kappa}{\Rightarrow} k\circ m \overset{\theta\circ m}{\Longrightarrow} g\circ m)$$

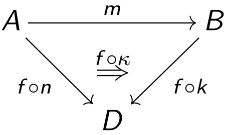
is a bijection. Such k is unique up to a unique isomorphism: write

$$k = \mathrm{lan}(m,n)\ .$$

In his thesis, Dubuc suggested "Lan" as a contraction of "left Kan".

Respecting left extensions

The left extension is *respected* by a morphism $f\colon C \to D$ when the diagram



exhibits $f \circ k$ as a left extension of $f \circ n$ along m; symbolically,

$$f \circ \mathrm{lan}(m, n) \cong \mathrm{lan}(m, f \circ n)\ .$$

Right adjoints as left extensions

Here is what Dubuc called "The Formal Adjoint Functor Theorem".

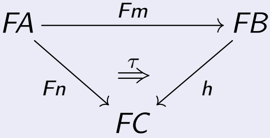
Proposition

A morphism $m\colon A \to B$ *in a bicategory has a right adjoint if and only if the identity of* A *has a left extension* $\mathrm{lan}(m, 1_A)$ *along* m *which is respected by* m*. In that case,* $m^* = \mathrm{lan}(m, 1_A)$ *is the right adjoint and it is respected by all morphisms* $f\colon A \to D$*; that is,* $\mathrm{lan}(m, f) = f \circ m^*$*.*

Creation of left extensions

Definition

A lax functor $F : \mathscr{N} \to \mathscr{M}$ creates left extensions *when, given morphisms* $m : A \to B$ *and* $n : A \to C$ *in* $\mathscr{N}$ *and a diagram*



in $\mathscr{M}$ *which exhibits* $h = \mathrm{lan}(Fm, Fn)$*, there exists a diagram that you all remember and isomorphism* $h \cong Fk$ *unique up to isomorphism with*

$$F\kappa = (Fn \xrightarrow{\tau} h \circ Fm \cong Fk \circ Fm \xrightarrow{F_2} F(k \circ m)) \; ;$$

moreover, the remembered diagram must exhibit $k = \mathrm{lan}(m, n)$.

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- Pseudofunctors which create left extensions reflect the existence of right adjoints.
- Left extensions in a one-object bicategory $\Sigma\mathscr{V}$ are internal right cohoms in the monoidal category $\mathscr{V}$. So, for a monoidal functor $U\colon \mathscr{W} \to \mathscr{V}$, to say $\Sigma U\colon \Sigma\mathscr{W} \to \Sigma\mathscr{V}$ creates left extensions is to say $U\colon \mathscr{W} \to \mathscr{V}$ creates right cohoms.

Hopf monoidal comonads

- Some references here are
  (i) [Bruguières-Lack-Virelizier: Advances 227(2) (2011)],
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- For a monoidal comonad $(D, \varepsilon \colon D \to 1, \delta \colon D \to D^2, D_0 \colon I \to DI, D_2 \colon DX \otimes DY \to D(X \otimes Y))$ on a monoidal category $\mathscr{V}$, the fusion map is the natural transformation with components

$$v_{Y,DX} = (DY \otimes DX \xrightarrow{1 \otimes \delta} DY \otimes D^2X \xrightarrow{D_2} D(Y \otimes DX)) \ .$$

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- Examples include tensoring $D = H \otimes -$ with a Hopf monoid H in a braided $\mathscr{V}$.

Coalgebras for Hopf monoidal comonads

While not made explicit in reference (ii) of the last frame, the constructions are there for the next result in which $D\text{-}\mathrm{Coalg}$ is the monoidal category of Eilenberg-Moore D-coalgebras.

Theorem

If D is a Hopf monoidal comonad on a monoidal category $\mathscr{V}$ then the underlying functor

$$\mathrm{U}\colon D\text{-}\mathrm{Coalg} \longrightarrow \mathscr{V}$$

creates cohoms.

Easy examples

Let DGAb denote the category of differential graded (that is, chain complexes of) abelian groups. The strong monoidal comonadic functors

$$\Sigma\colon \mathrm{GAb} \to \mathrm{Ab}\ ,\ \ \mathrm{U}\colon \mathrm{DGAb} \to \mathrm{GAb}\ \text{ and }\ \Sigma\colon \mathrm{DGAb} \to \mathrm{Ab}$$

are all Hopf monoidal comonadic.
Therefore they reflect dualizability.
The dualizable objects of Ab are of course the finitely generated free abelian groups. So, for example, a chain complex of the form

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\begin{bmatrix} 3 \\ -2 \end{bmatrix}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{bmatrix} 2 & 3 \end{bmatrix}} \mathbb{Z} \to 0 \to 0 \to \dots$$

has a dual in DGAb.

Creative change of base

In the situation of the last Theorem, put $\mathscr{W} = D\text{-}\mathrm{Coalg}$ and assume $\mathscr{V}$ is cocomplete and closed. Then we have the bicategory $\mathscr{V}\text{-}\mathrm{Mod}$ of $\mathscr{V}$-enriched categories and modules (or distributors or profunctors) between them. Also, we have $\mathscr{W}\text{-}\mathrm{Mod}$.

Theorem

If the right adjoint to U *preserves colimits, the change of base pseudofunctor*

$$\mathrm{U}_* \colon \mathscr{W}\text{-}\mathrm{Mod} \longrightarrow \mathscr{V}\text{-}\mathrm{Mod}$$

creates left extensions. In particular, a $\mathscr{W}$*-module* $M \colon \mathscr{K} \nrightarrow \mathscr{A}$ *is Cauchy if and only if the* $\mathscr{V}$*-module* $\mathrm{U}_*M \colon \mathrm{U}_*\mathscr{K} \nrightarrow \mathrm{U}_*\mathscr{A}$ *is.*

Incidentally, the right adjoints of all of $\Sigma \colon \mathrm{GAb} \to \mathrm{Ab}$, $\mathrm{U} \colon \mathrm{DGAb} \to \mathrm{GAb}$ and $\Sigma \colon \mathrm{DGAb} \to \mathrm{Ab}$ have further right adjoints.

Example

- A DG-module $M : \mathscr{I} \nrightarrow \mathscr{A}$ from the unit DG-category $\mathscr{I}$ to a small DG-category $\mathscr{A}$ amounts to a DG-functor $M \colon \mathscr{A}^{\mathrm{op}} \to \mathrm{DGAb}$.

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- The Cauchy completion $\mathscr{Q}\mathscr{A}$ of $\mathscr{A}$ (following Lawvere) is the full sub-DG-category of the presheaf DG-category $[\mathscr{A}^{\mathrm{op}}, \mathrm{DGAb}]$ consisting of those M which have a right adjoint module; that is, the DG-functor $[\mathscr{A}^{\mathrm{op}}, \mathrm{DGAb}](M, -)$ preserves small weighted colimits.

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- DG-Morita Theorem:

$$[\mathscr{A}^{\mathrm{op}}, \mathrm{DGAb}] \simeq [\mathscr{B}^{\mathrm{op}}, \mathrm{DGAb}] \text{ if and only if } \mathscr{Q}\mathscr{A} \simeq \mathscr{Q}\mathscr{B}$$

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- The Theorem implies that a DG-module $M : \mathscr{I} \nrightarrow \mathscr{A}$ is Cauchy if and only if the additive module $\Sigma_* M : \mathscr{I} \nrightarrow \Sigma_* \mathscr{A}$ is a retract of a finite direct sum of representables in the additive presheaf category on $\Sigma_* \mathscr{A}$.

The general context

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- We show that Caten admits the Eilenberg-Moore construction $\mathscr{V}^{\mathscr{G}}$ for comonads with underlying $\mathscr{U} : \mathscr{V}^{\mathscr{G}} \to \mathscr{V}$ actually a pseudofunctor. So

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- We adapt the Beck Comonadicity Theorem internally to Caten.
- We produce a fusion map v for any comonad $(\mathscr{V}, \mathscr{G})$ in Caten and define $\mathscr{G}$ to be left Hopf when v is invertible. Indeed, if $\mathscr{G}$ is Hopf so is the comonad generated by $\mathscr{U}_*$ and its right adjoint.

The general theorems

Theorem

If $\mathscr{G}$ is a left Hopf comonad on the bicategory $\mathscr{V}$ in Caten then the pseudofunctor $\mathscr{U} : \mathscr{V}^{\mathscr{G}} \to \mathscr{V}$ creates left extensions.

Theorem

If $\mathscr{G}$ is a comonad on the locally cocomplete bicategory $\mathscr{V}$ in Caten then the $\mathscr{U}$-induced pseudofunctor

$$\widetilde{\mathscr{U}} : \mathscr{V}^{\mathscr{G}}\text{-Mod} \to \mathscr{V}\text{-Mod}$$

is comonadic in CATEN via a comonad $\widetilde{\mathscr{G}}$ on $\mathscr{V}\text{-Mod}$. If $\mathscr{G}$ is left Hopf comonad and the right adjoint to $\mathscr{U}$ preserves local colimits then the comonad $\widetilde{\mathscr{G}}$ is also Hopf.

Thank You

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