VRIJE UNIVERSITEIT BRUSSEL

Hopf Categories

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Enriched category theory

$\mathcal{V} = (\mathcal{V}, \otimes, k)$ is a strict monoidal category, X is a class.
New monoidal category $(\mathcal{V}(X), \bullet, J)$

- An object is a family of objects M in $\mathcal{V}$ indexed by $X \times X$:

$$M = (M_{x,y})_{x,y \in X}.$$

- morphism $\varphi : \ M \to N$: family of morphisms $\varphi_{x,y} : \ M_{x,y} \to N_{x,y}$
- $(M \bullet N)_{x,y} = M_{x,y} \otimes N_{x,y}$, $J_{x,y} = ke_{x,y}$

functor $(-)^{\mathrm{op}} : \ \mathcal{V}(X) \to \mathcal{V}(X)$: $V^{\mathrm{op}}_{y,x} = V_{x,y}$, $\varphi^{\mathrm{op}}_{y,x} = \varphi_{x,y}$.

Enriched category theory

$\mathcal{V}$-category A

- class X
- multiplication morphisms $m = m_{x,y,z} : \ A_{x,y} \otimes A_{y,z} \to A_{x,z}$
- unit morphisms $\eta_x : \ J_{x,x} = ke_{x,x} \to A_{x,x}$

with unit and associativity conditions. J is a $\mathcal{V}$-category.

- $(\mathcal{V}, \otimes, k) = (\underline{\mathrm{Sets}}, \times, \{*\})$: ordinary categories
- $(\mathcal{V}, \otimes, k) = (\mathcal{M}_k, \otimes, k)$: k-linear categories

Enriched category theory

- If $\mathcal{V}$ is braided: tensor product in $\mathcal{V}(X)$ of two $\mathcal{V}$-categories is again a $\mathcal{V}$-category.
- Fix a class X: $\mathcal{V}$-X-categories; $\mathcal{V}$-X-functor is functor that is the identity on objects.

Semi-Hopf categories

Assume that $\mathcal{V}$ is braided.
$\underline{\mathcal{C}}(\mathcal{V})$ is the category of coalgebras in $\mathcal{V}$.
We consider $\underline{\mathcal{C}}(\mathcal{V})$-categories, aka semi-Hopf $\mathcal{V}$-categories.
Description
Coalgebra in $\mathcal{V}(X)$ is a family of coalgebras $(C_{x,y})$.
Structure maps: $\Delta_{x,y}: \ C_{x,y} \to C_{x,y} \otimes C_{x,y}$ and
$\varepsilon_{x,y}: \ C_{x,y} \to J_{x,y} = ke_{x,y}$

Semi-Hopf categories

Proposition

A semi-Hopf $\mathcal{V}$-category with underlying class X consists of $A \in \mathcal{V}(X)$ which is

- *a $\mathcal{V}$-category*
- *a coalgebra in $\mathcal{V}(X)$*
- *the morphisms $\Delta_{x,y}$ and $\varepsilon_{x,y}$ define $\mathcal{V}$-X-functors $\Delta : A \to A \bullet A$ and $\varepsilon : A \to J$.*

$\underline{\mathcal{C}}(\mathcal{V})$-categories with one object correspond to bialgebras in $\mathcal{V}$

op and cop

op

If A is a $\mathcal{V}$-category, then A^{op} is also a $\mathcal{V}$-category: multiplication morphisms

$$m^{\mathrm{op}}_{x,y,z} = m_{z,y,x} \circ c_{A_{y,x},A_{x,y}} : A^{\mathrm{op}}_{x,y} \otimes A^{\mathrm{op}}_{y,z} = A_{y,x} \otimes A_{z,y} \to A^{\mathrm{op}}_{x,z} = A_{z,x}$$

and unit morphisms $\eta^{\mathrm{op}}_x = \eta_x$.

If A is a $\underline{\mathcal{C}}(\mathcal{V})$-category, then A^{op} is also a $\underline{\mathcal{C}}(\mathcal{V})$-category, with coalgebra structure maps $\Delta^{\mathrm{op}}_{x,y} = \Delta_{y,x}$ and $\varepsilon^{\mathrm{op}}_{x,y} = \varepsilon_{y,x}$.

cop

Let C be a coalgebra in $\mathcal{V}(X)$. The coopposite coalgebra C^{cop} is equal to C as an object of $\mathcal{V}(X)$, with comultiplication maps

$$\Delta^{\mathrm{cop}}_{x,y} = c_{C_{x,y},C_{x,y}} \circ \Delta_{x,y} : \; C_{x,y} \to C_{x,y} \otimes C_{x,y},$$

and counit maps $\varepsilon_{x,y}$.

If A is a $\underline{\mathcal{C}}(\mathcal{V})$-category, then A^{cop} is also a $\underline{\mathcal{C}}(\mathcal{V})$-category; the $\mathcal{V}$-category structures on A and A^{cop} coincide.

Hopf categories

Definition

A Hopf $\mathcal{V}$-category is a semi-Hopf $\mathcal{V}$-category A together with a morphism $S: A \to A^{\mathrm{op}}$ in $\mathcal{V}(X)$ ($S_{x,y}: A_{x,y} \to A_{y,x}$) such that

$$\begin{aligned} m_{x,y,x} \circ (A_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y} &= \eta_x \circ \varepsilon_{x,y} : A_{x,y} \to A_{x,x}; \\ m_{y,x,y} \circ (S_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y} &= \eta_y \circ \varepsilon_{x,y} : A_{x,y} \to A_{y,y}, \end{aligned}$$

for all $x, y \in X$.

Over $\mathcal{M}_k$: for $h \in A_{x,y}$:

$$h_{(1)} S_{x,y}(h_{(2)}) = \varepsilon_{x,y}(h) 1_x \quad ; \quad S_{x,y}(h_{(1)}) h_{(2)} = \varepsilon_{x,y}(h) 1_y.$$

A Hopf $\mathcal{V}$-category with one object is a Hopf algebra in $\mathcal{V}$.

Hopf-categories and groupoids

$\mathcal{V} = (\underline{\mathrm{Sets}}, \times, \{*\})$.
Every set is in a unique way a coalgebra in $\underline{\mathrm{Sets}}$.
$\mathcal{C}(\underline{\mathrm{Sets}}) = \underline{\mathrm{Sets}}$. $\mathcal{C}(\underline{\mathrm{Sets}})$-categories = categories.

Proposition

A Hopf $\underline{\mathrm{Sets}}$-category is the same thing as a groupoid (i.e. a category in which all morphisms are isomorphisms).

Hopf-categories: first properties

Theorem

Let A be a Hopf $\mathcal{V}$-category. The antipode S is a morphism of $\underline{\mathcal{C}}(\mathcal{V})$-categories $H \to H^{\mathrm{opcop}}$.

Proposition

Let A be a k-linear Hopf category. For $x, y \in X$, the following assertions are equivalent.

1. $S_{x,y}(h_{(2)})h_{(1)} = \varepsilon_{x,y}(h)1_y$, *for all* $h \in A_{x,y}$;
2. $h_{(2)}S_{x,y}(h_{(1)}) = \varepsilon_{x,y}(h)1_x$, *for all* $h \in A_{x,y}$;
3. $S_{y,x} \circ S_{x,y} = A_{x,y}$.

Hopf-categories: first properties

Let A and B be Hopf $\mathcal{V}$-categories. A $\underline{\mathcal{C}}(\mathcal{V})$-functor $f : A \to B$ is called a Hopf $\mathcal{V}$-functor if

$$S^B_{f(x),f(y)} \circ f_{x,y} = f_{y,x} \circ S^A_{x,y}, \tag{1}$$

for all $x, y \in X$.

Proposition

Let A and B be Hopf $\mathcal{V}$-categories. If $f : A \to B$ is a $\underline{\mathcal{C}}(\mathcal{V})$-functor, then it is also a Hopf $\mathcal{V}$-functor.

The representation category

Let A be a $\mathcal{V}$-category. A left A-module is an object M in $\mathcal{V}(X)$ together with a family of morphisms in $\mathcal{V}$

$$\psi = \psi_{x,y,z} : \; A_{x,y} \otimes M_{y,z} \to M_{x,z}$$

$+$ associativity and unit conditions.
A morphism $\varphi : \; M \to N$ in $\mathcal{V}(X)$ between left A-modules is called left A-linear if

$$\varphi_{x,z} \circ \psi_{x,y,z} = \psi_{x,y,z} \circ (A_{x,y} \otimes \varphi_{y,z})$$

Category: ${}_A\mathcal{V}(X)$

The representation category

Proposition

Let A be a $\mathcal{C}(\mathcal{V})$-category. Then there is a monoidal structure on ${}_A\mathcal{V}(X)$ such that the forgetful functor ${}_A\mathcal{V}(X) \to \mathcal{V}(X)$ is monoidal.

Bewijs.

(in case $\mathcal{V} = \mathcal{M}_k$). We need actions

$$A_{x,y} \otimes M_{y,z} \otimes N_{y,z} \to M_{x,z} \otimes N_{x,z} \quad \text{and} \quad A_{x,y} \otimes ke_{y,z} \to ke_{x,z}.$$

Take

$$a \cdot (m \otimes n) = a_{(1)} m \otimes a_{(2)} n \quad \text{and} \quad a \cdot 1 = \varepsilon(a).$$

□

Duality: $\mathcal{V}$-opcategories

Hopf categories and Hopf group (co)algebras

Hopf categories and weak Hopf algebras

Proposition

Let A be a k-linear Hopf category, with $|A| = X$ a finite set. Then $A = \oplus_{x,y\in X} A_{x,y}$ is a weak Hopf algebra.

Example

Take a groupoid with finitely many objects; apply the linearization functor to obtain a k-linear Hopf category; in packed form it becomes the groupoid algebra, which is well-known to be a weak Hopf algebra.

Proposition

Let C be a k-linear Hopf opcategory, with $|C| = X$ a finite set. Then $C = \oplus_{x,y\in X} C_{x,y}$ is a weak Hopf algebra.

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- G. Böhm, Y. Chen, L. Zhang, “On Hopf monoids in duoidal categories”, *J. Algebra* **394** (2013), 139-172.

Hopf categories and duoidal categories

Definition

A duoidal category is a category $\mathcal{M}$ with

- monoidal structure $(\odot, I)$
- monoidal structure $(\bullet, J)$
- $\delta : \ I \to I \bullet I$
- $\varpi : \ J \odot J \to J$
- $\tau : I \to J$
- $\zeta_{A,B,C,D} : (A \bullet B) \odot (C \bullet D) \to (A \odot C) \bullet (B \odot D)$
- (J, ϖ, τ) is an algebra in $(\mathcal{M}, \odot, I)$
- (I, δ, τ) is a coalgebra in $(\mathcal{M}, \bullet, J)$
- 6 more commutative diagrams (2 associativity and 4 unit)

Hopf categories and duoidal categories

Let X be a set. $(\mathcal{M}_k(X), \bullet, J)$ is a monoidal category.
Second monomial structure:

$$(M \odot N)_{x,z} = \oplus_{y \in X} M_{x,y} \otimes N_{y,z}.$$

$$I_{x,y} = \begin{cases} ke_{x,x} & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

- $\tau : \ I \to J$: natural inclusion
- $\delta : \ I \to I \bullet I = I$: identity map
- $(J \odot J)_{x,y} = \oplus_{z \in X} ke_{x,z} \otimes ke_{z,y} = \oplus_{z \in X} kze_{x,y} = kXe_{x,y}.$
  $\varpi : \ J \odot J \to J$
  $\varpi_{x,y} : \ \oplus_{z \in X} kze_{x,y} \to ke_{x,y}$
  $\varpi_{x,y}(\sum_{z \in X} \alpha_z ze_{x,y}) = \sum_{z \in X} \alpha_z e_{x,y}.$

$$((M \bullet N) \odot (P \bullet Q))_{x,y} = \bigoplus_{z \in X} M_{x,z} \otimes N_{x,z} \otimes P_{z,y} \otimes Q_{z,y};$$

$$((M \odot P) \bullet (N \odot Q))_{x,y} = \bigoplus_{u,v \in X} M_{x,u} \otimes P_{u,y} \otimes N_{x,v} \otimes Q_{v,y},$$

$\zeta_{M,N,P,Q,x,y}$ is the map switching the second and third tensor factor, followed by the natural inclusion.

Theorem

Let X be a set. $(\mathcal{M}_k(X), \odot, I, \bullet, J, \delta, \varpi, \tau, \zeta)$ is a duoidal category.

Definition

Let $(\mathcal{M}, \odot, I, \bullet, J, \delta, \varpi, \tau, \zeta)$ be a duoidal category. A bimonoid is an object A, together with an algebra structure (μ, η) in $(\mathcal{M}, \odot, I)$ and a coalgebra structure (Δ, ε) in $(\mathcal{M}, \bullet, J)$ subject to the compatibility conditions

$$
\begin{aligned}
\Delta \circ \mu &= (\mu \bullet \mu) \circ \zeta \circ (\Delta \odot \Delta); \\
\varpi \circ (\varepsilon \odot \varepsilon) &= \varepsilon \circ \mu; \\
(\eta \bullet \eta) \circ \delta &= \Delta \circ \eta; \\
\varepsilon \circ \eta &= \tau.
\end{aligned}
$$

Hopf categories and duoidal categories

Theorem

Let X be a set, and let $A \in \mathcal{M}_k(X)$. We have a bijective correspondence between bimonoid structures on A over the duoidal category $(\mathcal{M}_k(X), \odot, I, \bullet, J, \delta, \varpi, \tau, \zeta)$ from and k-linear semi-Hopf category structures on A.

Hopf modules

Definition

A is a k-linear semi-Hopf category. A Hopf module over A is $M \in \mathcal{M}_k(X)$ such that

- $M \in \mathcal{M}_k(X)_A$, with structure maps $\psi_{x,y,z}$
- $M \in \mathcal{M}_k(X)^A$: M is a right comodule over A as a coalgebra in $\mathcal{M}_k(X)$, with structure maps $\rho_{x,y}$
- $\rho_{x,z}(ma) = m_{[0]}a_{(1)} \otimes m_{[1]}a_{(2)}$

Category of Hopf modules: $\mathcal{M}_k(X)^A_A$.
New category: $\mathcal{D}(X)$ consisting of families of k-modules $N = (N_x)_{x \in X}$ indexed by X.

An adjoint pair of functors

Proposition

We have a pair of adjoint functors (F, G) between the categories $\mathcal{D}(X)$ and $\mathcal{M}_k(X)_A^A$.

Bewijs.

$F(N)_{x,y} = N_x \otimes A_{x,y}$, with

$$(n \otimes a)b = n \otimes ab \quad ; \quad \rho_{x,y}(n \otimes a) = n \otimes a_{(1)} \otimes a_{(2)},$$

$G(M) = M^{\mathrm{co}A} \in \mathcal{D}(X)$ is given by the formula

$$M_x^{\mathrm{co}A} = M_{x,x}^{\mathrm{co}A_{x,x}} = \{m \in M_{x,x} \mid \rho_{x,x}(m) = m \otimes 1_x\}.$$

$\square$

The fundamental theorem

Canonical maps:

$$\mathrm{can}^z_{x,y} : \; A_{z,x} \otimes A_{x,y} \to A_{z,y} \otimes A_{x,y}, \quad \mathrm{can}^z_{x,y}(a \otimes b) = ab_{(1)} \otimes b_{(2)}.$$

Theorem

For a k-linear semi-Hopf category A with underlying class X, the following assertions are equivalent.

1. *A is a k-linear Hopf category;*
2. *the pair of adjoint functors (F, G) is a pair of inverse equivalences between the categories $\mathcal{D}(X)$ and $\mathcal{M}_k(X)^A_A$;*
3. *the functor G is fully faithful;*
4. $\mathrm{can}^z_{x,y}$ *is an isomorphism, for all $x, y, z \in X$;*
5. $\mathrm{can}^x_{x,y}$ *and* $\mathrm{can}^y_{x,y}$ *are isomorphisms, for all $x, y \in X$.*

Applications

Proposition

Let A be a Hopf category in $\mathcal{M}_k^{\mathrm{f}}(X)$. Then A^* is a Hopf module.
$\rho_{x,y}:\ A^*_{x,y} \to A^*_{x,y} \otimes A_{x,y}$:

$$\rho_{x,y}(a^*) = \sum_i a^* a_i^* \otimes a_i$$

$\psi_{x,y,z}:\ A^*_{x,y} \otimes A_{y,z} \to A^*_{x,z}$:

$$\langle a^* \leftharpoonup a, b\rangle = \langle a^*, bS_{y,z}(a)\rangle$$

$$\begin{aligned} A_x^{*\mathrm{co}A} &= (A^*_{x,x})^{\mathrm{co}A_{x,x}} = \textstyle\int^l_{A^*_{x,x}} \\ &= \{\varphi \in A^*_{x,x} \mid \varphi a^* = \langle a^*, 1_x\rangle\varphi,\ \text{for all } a^* \in A^*_{x,x}\} \end{aligned}$$

is the space of left integrals on $A_{x,x}$.

Applications

Corollary

For a semi-Hopf category in $\mathcal{M}_k^{\mathrm{f}}(X)$,

$$\alpha_{x,y} = \varepsilon_{x,y}^{A^*} : \int_{A^*_{x,x}}^{l} \otimes A_{x,y} \to A^*_{x,y}, \quad \varepsilon_{x,y}^{A^*}(\varphi \otimes a) = \varphi \leftharpoonup a.$$

is an isomorphism, for all x, y.

Proposition

Let A be a Hopf category in $\mathcal{M}_k^{\mathrm{f}}(X)$. The antipode maps $S_{x,y} : A_{x,y} \to A_{y,x}$ are bijective, for all $x, y \in X$.

Hopf-Galois theory

Let H be k-linear Hopf category. A right H-comodule category consists of

- k-linear category A
- A_{xy} is a right H_{xy}-comodule
- $\rho_{xz}(ab) = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]}$, for $a \in A_{xy}$ and $b \in A_{yz}$
- $\rho_{xx}(1_x^A) = 1_x^A \otimes 1_x^H$

$B = A^{\mathrm{co}H}$
Canonical maps:

$$\mathrm{can}^z_{xy} : A_{zx} \otimes_{B_x} A_{xy} \to A_{zy} \otimes H_{xy}, \quad \mathrm{can}^z_{xy}(a \otimes a') = aa'_{[0]} \otimes a'_{[1]}.$$

If these are isomorphisms: A is H-Galois extension of B.

Hopf-Galois theory: further observations

- Under appropriate flatness assumptions: H-Galois condition gives structure theorem for relative Hopf modules
- Our theory involves coactions by Hopf category (as in Chase-Sweedler); in finite case, one passes to the dual, to get actions by the dual Hopf **op**category. This works
- Paques and Tamusianas (A Galois-Grothendieck-type correspondence for groupoid actions, Algebra Discr. Math. 17 (2014), 80-97) develop Galois theory for **actions** by groupoids. It does not fit into our picture

Larson-Sweedler Theorem

Theorem

A finite dimensional Hopf algebra over a field is a Frobenius algebra.

Buckley, Fieremans, Vasilkaopoulou and Vercruysse bring the appropriate generalization to Hopf $\mathcal{V}$-categories.

Larson-Sweedler Theorem

Definition

A Frobenius $\mathcal{V}$-category is a $\mathcal{V}$-category that is also a $\mathcal{V}$-opcategory such that

$$\begin{tikzcd}
A_{x,y} \otimes A_{y,z} \arrow[rr, "d_{x,w,y} \otimes 1"] \arrow[dr, "m_{x,y,z}"] \arrow[dd, "1 \otimes d_{y,w,z}"'] & & A_{x,w} \otimes A_{w,y} \otimes A_{y,z} \arrow[dd, "1 \otimes m_{w,y,z}"] \\
& A_{x,z} \arrow[dr, "d_{x,w,z}"] & \\
A_{x,y} \otimes A_{y,w} \otimes A_{w,z} \arrow[rr, "m_{x,y,w} \otimes 1"'] & & A_{x.w} \otimes A_{w,z}
\end{tikzcd}$$

commutes.

References

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- M. Buckley, T. Fieremans, C. Vasilakopoulou, J. Vercruysse, A Larson-Sweedler Theorem for Hopf $\mathcal{V}$-categories, in progress.