Stone Representation Theorem for Boolean Algebras in the Topos of (Pre)Sheaves on a Monoid

S. Sepahani M. Mahmoudi

Department of Mathematics
Shahid Beheshti University Tehran

CT, 13 July 2018

Previously Done

Boolean Algebras in a localic topos
Banaschewski, Bhutani; 1986
Borceux, Peddicchio, Rossi; 1990

The Category **MSet**

- **MSet** $\simeq$ **Set**^{M}
- Limits as in **Set**
- The subobject classifier $\Omega = \{K | \text{K is a left ideal of M}\}$
  - $mK = \{x \in M | xm \in K\}$
- Exponentiation $B^A = \{f | f : M \times A \to B : \text{f is equivariant}\} = \{f | f = (f_s) : \forall s, t \in M, f_s : A \to B, tf_s = f_{ts}t\}$
- Free functor F : **Set** $\to$ **MSet**: $F(X) = M \times X$
  $m(n, x) = (mn, x)$
- Cofree functor H : **Set** $\to$ **MSet**: $H(X) = \{f : M \to X\}$
  $(mf)(n) = f(nm)$
  - $H(2) = \mathcal{P}(M), mX = \{x \in M | xm \in X\}$
  - H : **Boo** $\to$ **MBoo**
- Monomorphisms in **MSet** are equivariant one-one maps

Closure Operator in a Category

Definition

A family $C = (C_X)_{X \in MSet}$, with $C_X : Sub(X) \to Sub(X)$ taking $Y \leq X$ to $C_X(Y)$, is called a closure operator on MSet if it satisfies the following:

1. (Extension) $Y \leq C_X(Y)$
2. (Monotonicity) $Y_1 \leq Y_2 \Rightarrow C_X(Y_1) \leq C_X(Y_2)$
3. (Continuity) $f(C_X(Y)) \leq C_Z(f(X))$ for all morphisms $f : X \to Z$

and we say that C is idempotetnt if additionally we have $C_X(C_X(Y)) = Y$ for every $Y \leq X$

for $Y \leq X$, Y is said to be

- closed in X if $C_X(Y) = Y$
- dense in X if $C_X(Y) = X$

Ideal Closure Operator for $I \trianglelefteq_R M$

Definition

Let $A \hookrightarrow B$. $C^I(A) = \{b \in B | \forall s \in I, sb \in A\}$

- C^I is idempotent iff I is idempotent

Ideal Closure Operator for $I \trianglelefteq_R M$

Definition

Let $A \hookrightarrow B$. $C^I(A) = \{b \in B | \forall s \in I, sb \in A\}$

- C^I is idempotent iff I is idempotent
- $j^I(K) = \{x \in M | \forall s \in I, sx \in K\}$

Ideal Closure Operator for $I \trianglelefteq_R M$

Definition

Let $A \hookrightarrow B$. $C^I(A) = \{b \in B | \forall s \in I, sb \in A\}$

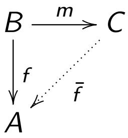
- C^I is idempotent iff I is idempotent
- $j^I(K) = \{x \in M | \forall s \in I, sx \in K\}$
- $m : Y \rightarrowtail X$ is I-dense if $\forall s \in I, \forall x \in X, sx \in Y$

I-Separated Objects and I-Sheaves

$A \in \mathbf{MSet}$ is an I-separated object if for every dense monomorphism m, any two equivariant maps from C to A making the diagram commutative are equivalent. A is an I-sheaf if this map uniquely exists for every I-dense m and every f.

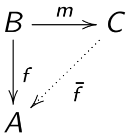
I-Separated Objects and I-Sheaves

$A \in$ **MSet** is an I-separated object if for every dense monomorphism m, any two equivariant maps from C to A making the diagram commutative are equivalent. A is an I-sheaf if this map uniquely exists for every I-dense m and every f.



I-Separated Objects and I-Sheaves

$A \in \mathbf{MSet}$ is an I-separated object if for every dense monomorphism m, any two equivariant maps from C to A making the diagram commutative are equivalent. A is an I-sheaf if this map uniquely exists for every I-dense m and every f.



Remark

A is I-separated iff $\forall a, b \in A, (\forall s \in I, sa = sb \Rightarrow a = b)$

The Category $\mathcal{S}h_{j^l}$**MSet**

- $\mathcal{S}h_{j^l}$**MSet** is closed under limits in **MSet**.
- $\mathcal{S}h_{j^l}$**MSet** is closed under exponentiation in **MSet**.
- $\Omega_{j^l} = Eq(j^l, id_\Omega)$ is the subobject classifier of $\mathcal{S}h_{j^l}$**MSet**
  - $\Omega_{j^l} \leq im(j^l)$

The Category $\mathcal{S}h_{j^I}$**MSet**

- $\mathcal{S}h_{j^I}$**MSet** is closed under limits in **MSet**.
- $\mathcal{S}h_{j^I}$**MSet** is closed under exponentiation in **MSet**.
- $\Omega_{j^I} = Eq(j^I, id_\Omega)$ is the subobject classifier of $\mathcal{S}h_{j^I}$**MSet**
  - $\Omega_{j^I} \leq im(j^I)$

Sh_{j^I}**MSet** is a topos.

$\mathcal{S}h_{j'}$**MSet** is a subtopos of **MSet**

Theorem

(Adamek, Herrlich, Strecker) If $\mathcal{E}$ is strongly complete and co-wellpowered, then the following conditions are equivalent for any functor $G : \mathcal{E} \to \mathcal{F}$:

- G is adjoint
- G preserves small limits and is cowellpowered.

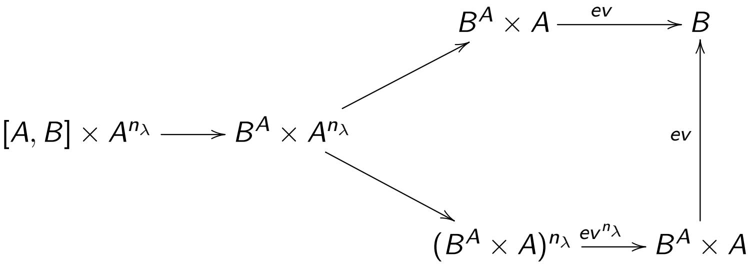
Proposition

(Johnstone) Let $\mathcal{E}$ be a cartesian closed category, and $\mathcal{L}$ be a reflective subcategory of $\mathcal{E}$, corresponding to a reflector L on $\mathcal{E}$. Then $\mathcal{L}$ preserves finite products iff $\mathcal{L}$ is an exponential ideal of $\mathcal{E}$.

Boolean Algebras in a Topos

- **MBoo**
- $\mathcal{S}h_{j'}$**Boo**
- H : **Set** $\to$ **MSet** can be lifted to H : **Boo** $\to$ **MBoo**
- An internal counterpart for $Ult(A)$ for a Boolean algebra A.

Internal hom Object



Internal hom Object in **MBoo** and in $Boo\mathcal{S}h_{j^l}$**MSet**

Definition

In $Boo\mathcal{S}h_{j^l}MSet$ we have the following explicit definition for $[A, B]$
$[A, B] = \{(f_s)_{s\in M} | \text{for every } s \in M, f_s : A \to B \text{ is a Boolean homomorphism}, \forall s, t \in M, tf_s = f_{ts}t\}$

Example

$f : A \to B$ Boolean homomorphism for $A, B \in MSet$. Let $f_e = f$ and for every $s \in M$, $f_s = sfs^{-1}$. Then $(f_s)_{s\in M} \in [A, B]$.

Initial Boolean Algebras

In **Set**

The initial Boolean algebra is 2, the two-element Boolean algebra.

In **MSet**

The initial Boolean algebra is **2**. i.e. The two-element Boolean algebra with identity action of M.

in $BooSh_{j^I}$ **MSet**

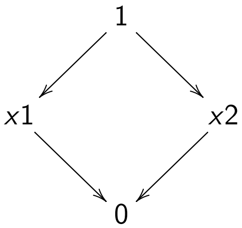
The initial Boolean algebra is the sheaf reflection of **2** which is the I-closure of **2** in $\Omega_{j^I}^{\mathbf{2}}$:

$$\bar{\mathbf{2}} = \{f \in \Omega_{j^I}^{\mathbf{2}} : \forall s \in I, sf \in \mathbf{2}\}$$

Example

*	e	a	b
e	e	a	b
a	a	a	b
b	b	a	b

$$I = \{a, b\}$$



Ore-like Conditions

Lemma

If for the monoid M and its right ideal I we have that

$$\exists s \in I \forall t \in M, Ms \cap Mst \neq \varnothing$$

then $\mathbf{2}$ is injective with respect to all I-dense monomorphisms and $\bar{\mathbf{2}} = \mathbf{2}$

Lemma

If for the monoid M and its right ideal I we have that $2 = \bar{2}$ then

$$\forall t \in M, Mt \cap MI \neq \varnothing$$

Stone Map in **Set**

Lemma

The functor $\mathcal{U}lt(-) : \mathbf{Boo} \to \mathbf{Set}$ is left adjoint to the functor $\mathcal{P}(-) : \mathbf{Set} \to \mathbf{Boo}$.

$s : A \to \mathcal{P}(\mathcal{U}lt(A))$ is the unit of the adjunction at A. $s(a)(\alpha) = \alpha(a)$.

$$\begin{array}{ccc} \mathcal{P}(\mathcal{U}lt(A)) \times \mathcal{U}lt(A) & \longrightarrow & \mathbf{2} \\ {\scriptstyle s(a)\times id_{\mathcal{U}lt(A)}} \uparrow & \nearrow {\scriptstyle f} & \\ A \times \mathcal{U}lt(A) & & \end{array}$$

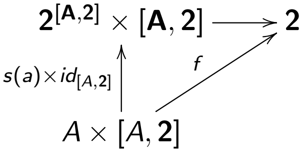
$$f(a, \alpha) = \alpha(a)$$

Stone Map in **MSet**

Lemma

The functor $[-, \mathbf{2}] : \mathbf{MBoo} \to \mathbf{MSet}$ is left adjoint to the functor $\mathbf{2}^{(-)} : \mathbf{MSet} \to \mathbf{MBoo}$.

Let $s : A \to \mathbf{2}^{[A,\mathbf{2}]}$ be the unit of the adjunction at A: $A \to \mathbf{2}^{[A,\mathbf{2}]}$. i.e. $s(a)(m, \alpha) = \alpha_e(ma)$.



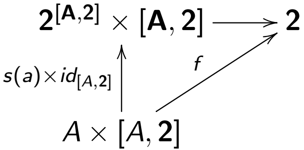
$$f(a, \alpha) = \alpha(e, a) = \alpha_e(a)$$

Stone Map in **MSet**

Lemma

The functor $[-, \mathbf{2}] : \mathbf{MBoo} \to \mathbf{MSet}$ is left adjoint to the functor $\mathbf{2}^{(-)} : \mathbf{MSet} \to \mathbf{MBoo}$.

Let $s : A \to \mathbf{2}^{[A,\mathbf{2}]}$ be the unit of the adjunction at A: $A \to \mathbf{2}^{[A,\mathbf{2}]}$. i.e. $s(a)(m, \alpha) = \alpha_e(ma)$.



$$f(a, \alpha) = \alpha(e, a) = \alpha_e(a)$$

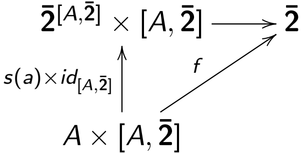
s is an embedding iff $\forall a \neq b \in A, \exists (m, \alpha) \in M \times [A, 2]$ s.t. $s(a)(m, \alpha) \neq s(b)(m, \alpha)$ or equivalently $\alpha_e(ma) \neq \alpha_e(mb)$

Stone Map in $\mathcal{S}h_{j'}\mathbf{MSet}$

Lemma

The functor $[-,\bar{\mathbf{2}}] : BooSh_{j'}\mathbf{MSet} \to Sh_{j'}\mathbf{MSet}$ is left adjoint to the functor $\bar{\mathbf{2}}^{(-)} : Sh_{j'}\mathbf{MSet} \to BooSh_{j'}\mathbf{MSet}$.

Let $s : A \to \bar{\mathbf{2}}^{[A,\bar{\mathbf{2}}]}$ be the unit of the adjucntion at A: $A \to \bar{\mathbf{2}}^{[A,\{\bar{\mathbf{2}}]}$. i.e. $s(a)(m,\alpha) = \alpha_e(ma)$.



$$f(a,\alpha) = \alpha(e,a)$$

When does Stone Representation Theorem hold in **MSet**?

Theorem

For a monoid M T.F.A.E.

- *s is an embedding for every $A \in MBoo$;*
- *s is an embedding for $H(\mathbf{2})$;*
- *M is a group.*

Summary

- The Stone Representation Theorem holds in **MBoo** iff **MSet** is Boolean.
- Still to be done
  - When is the Stone map an embedding in $BooSh_{j^!}MSet$?

Definition

$(X, \mathcal{T})$ a topological space object. $X \in \mathbf{MSet}$, $\mathcal{T} \leq \Omega^X$

-
  - $f_\varnothing \in \mathcal{T}$
  - $f_M \in \mathcal{T}$
  - for every index set I, if $\forall i \in I, f_i \in \mathcal{T}$ then $\bigvee_{i \in I} f_i \in \mathcal{T}$
  - for every finite index set I, if $\forall i \in I, f_i \in \mathcal{T}$ then $\bigwedge_{i \in I} f_i \in \mathcal{T}$

so we have a compatible family of topologies.

Define a Stone space in $MSet$ and in $Sh_{j^!}MSet$. (Neighborhood, zero-dimensionality, Hausdorffness,...)

- Axiom of choice

References I

- B. Banaschewski, K. R. Bhutani, Boolean algebras in a localic topos, *Math. Proc. Cambridge Philos. Soc.*, 100 (1986) 43–55.
- F. Borceux, M. C. Pedicchio, F. Rossi, Boolean algebras in a localic topos, *J. Pure Appl. Algebra* 68 (1990) 55–65.
- M. M. Ebrahimi, Algebras in a topos of sheaves, Ph.D. Thesis, *McMaster University*, 1980.
- M. M. Ebrahimi, Internal completeness and injectivity of Boolean algebras in the topos of M-Sets, *Bull. Aust. Math. Soc.* 41 (1990) 323–332.
- M. M. Ebrahimi, On ideal closure operators of M-Sets, *Southeast Asian Bull. Math.* 30 (3) (2006) 439–444.
- P. T. Johnstone, Sketches of an Elephant: A Topos Theory Compendium, Oxford Logic Guides, *Clarendon Press*, 2002.