On split extensions of bialgebras[1]

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[1] Joint work with Xabier García-Martínez.

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- Strategy: understand *split extensions*, which give us *protomodularity*.
- We first sketch the context where we shall be working.

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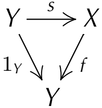

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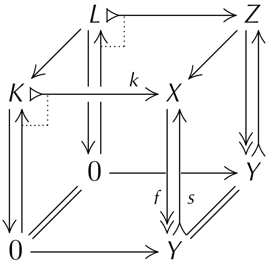

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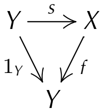

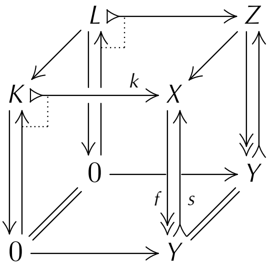

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For the sake of this talk, a **split extension** (f, s, k) is a point (f, s) with $k = \ker(f)$.

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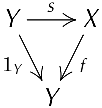
Protomodularity

A Barr-exact category is **semi-abelian** when it is pointed, has binary sums and is **protomodular**: the *Split Short Five Lemma* holds [Bourn, 1991].

This definition [Janelidze, Márki & Tholen, 2002] unifies "old" approaches towards an axiomatisation of categories "close to *Gp*" such as [Higgins, 1956] and [Huq, 1968] with "new" categorical algebra—the concepts of Barr-exactness and Bourn-protomodularity.

Examples: *Gp*, varieties of Ω-groups, $\mathit{Lie}_{\mathbb{K}}$, $\mathit{Alg}_{\mathbb{K}}$, *XMod*, *Loop*, $\mathit{HopfAlg}_{\mathbb{K},coc}$, C^*-*Alg*, $\mathit{Set}_*^{\mathrm{op}}$.

A **point** (f, s) **over** Y is a split epimorphism $f\colon X \to Y$ with a chosen splitting $s\colon Y \to X$.



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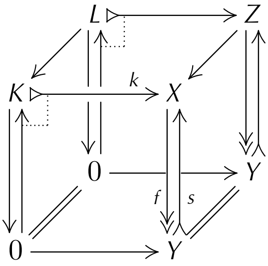

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So in a protomodular category, we can't see the difference. **Is $\mathit{HopfAlg}_{\mathbb{K}}$ protomodular?**

Split extensions and strong points

Given a split extension $0 \longrightarrow K \rightarrowtail^{k} X \underset{f}{\overset{s}{\leftrightarrows}} Y \cdots\!\!> 0$

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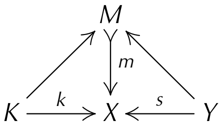

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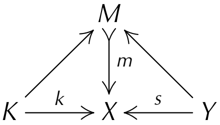

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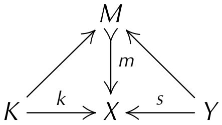

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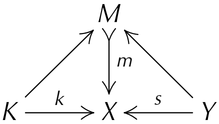

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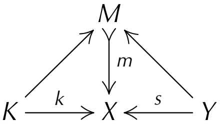

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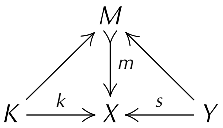

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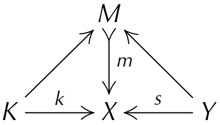

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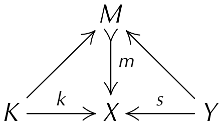

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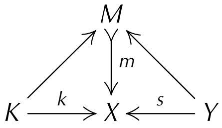

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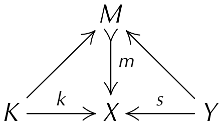

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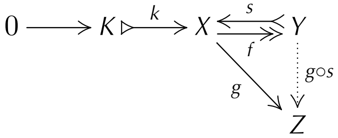

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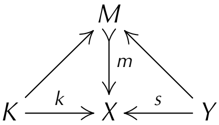

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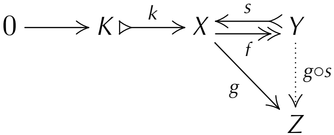

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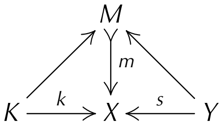
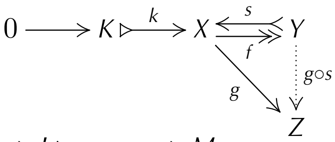
Two proofs



A pair (k, s) as on the left is **jointly extremally epimorphic** when m mono implies m iso.

A point (f, s) is **strong** when for $k = \ker(f)$, the pair (k, s) is jointly extremally epimorphic.

- The split epimorphism in a strong point is always the cokernel of its kernel.



Proof. Consider a strong point (f, s) and g such that $g \circ k = 0$. k and s are jointly epic, so $g \circ s \circ f \circ k = 0 = g \circ k$ and $g \circ s \circ f \circ s = g \circ s$ together imply $g \circ s \circ f = g$. □

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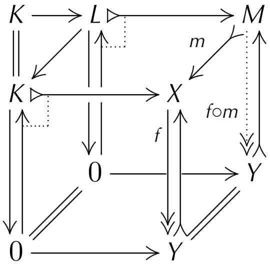

Proof. The arrow $L \to K$ is both a monomorphism and a split epimorphism, hence it is an isomorphism. m is then iso by the Split Short Five Lemma. □

Protomodular objects in $BiAlg_{\mathbb{K},coc}$

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Theorem [García-Martínez & VdL, 2017]

$\mathbb{K}$ an algebraically closed field.

For a cocommutative $\mathbb{K}$-bialgebra B, TFAE:

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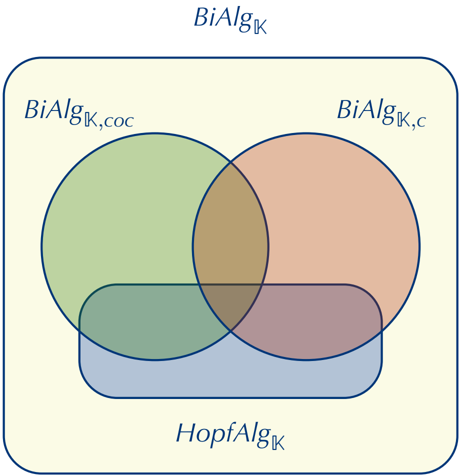

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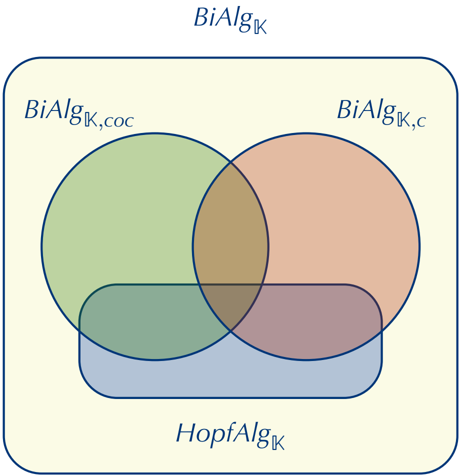

Does this extend to all Hopf algebras?

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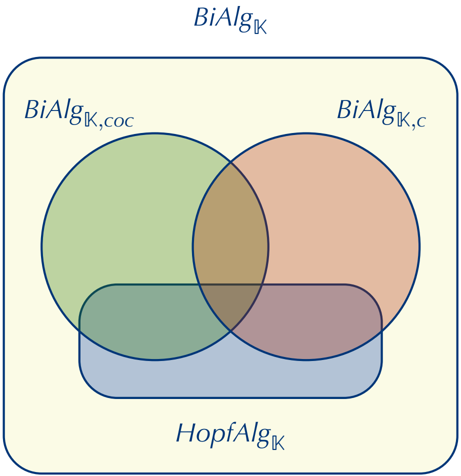

Does this extend to all Hopf algebras?
…to the commutative ones?

Beyond cocommutativity: when is a bialgebra protomodular?

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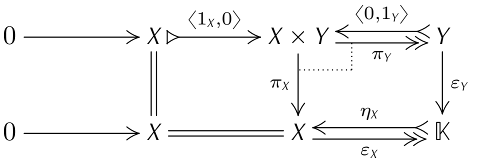

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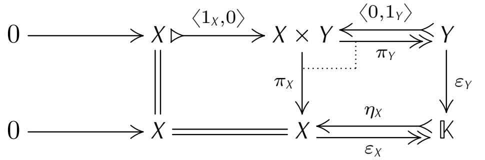

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Proof of the proposition

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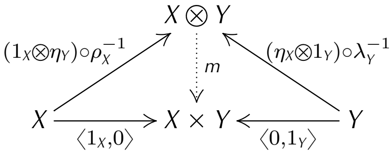

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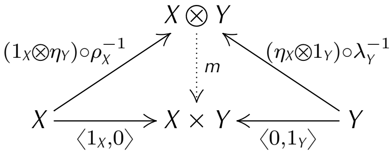

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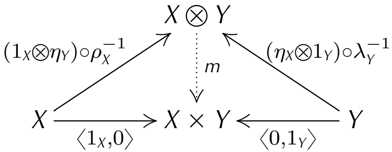

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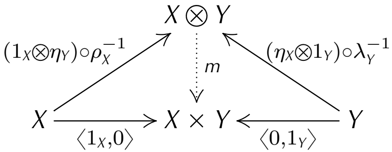

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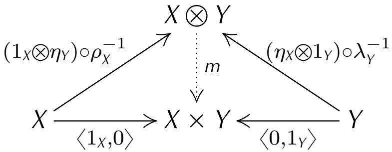

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$$\begin{aligned}
&(f \otimes g) \circ \Delta_Z \\
&= (\rho_X \otimes \lambda_Y) \circ (1_X \otimes \varepsilon_Y \otimes \varepsilon_X \otimes 1_Y) \circ (h \otimes h) \circ \Delta_Z \\
&= (\rho_X \otimes \lambda_Y) \circ (1_X \otimes \varepsilon_Y \otimes \varepsilon_X \otimes 1_Y) \circ \Delta_{X \otimes Y} \circ h \\
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&= (\rho_X \otimes \lambda_Y) \circ (1_X \otimes \varepsilon_X \otimes \varepsilon_Y \otimes 1_Y) \circ (\Delta_X \otimes \Delta_Y) \circ h \\
&= (\rho_X \otimes \lambda_Y) \circ (\rho_X^{-1} \otimes \lambda_Y^{-1}) \circ h = h.
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Here $\Delta_{X \otimes Y} \circ h = (h \otimes h) \circ \Delta_Z$ since h is a coalgebra morphism.

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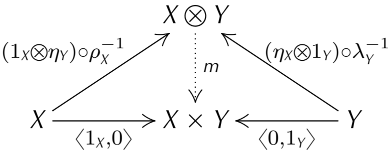


commute.

We prove that m is mono:
then Y unital implies that m is an isomorphism.

For $h \colon Z \to X \otimes Y$ in $BiAlg_{\mathbb{K}}$, we write

$$f = \rho_X \circ (1_X \otimes \varepsilon_Y) \circ h \colon Z \to X$$
$$g = \lambda_Y \circ (\varepsilon_X \otimes 1_Y) \circ h \colon Z \to Y$$

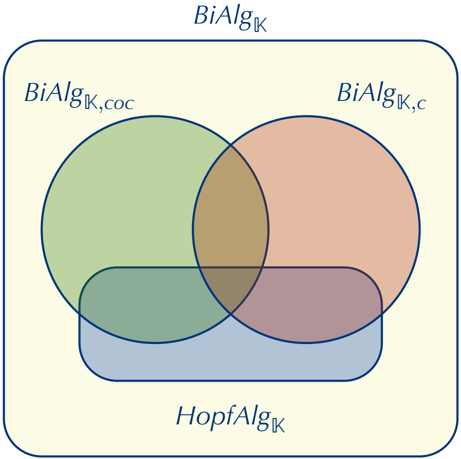
so that $\langle f, g \rangle = m \circ h$, and we prove that $h = (f \otimes g) \circ \Delta_Z$ in $Vect_{\mathbb{K}}$:

$$\begin{aligned}
&(f \otimes g) \circ \Delta_Z \\
&= (\rho_X \otimes \lambda_Y) \circ (1_X \otimes \varepsilon_Y \otimes \varepsilon_X \otimes 1_Y) \circ (h \otimes h) \circ \Delta_Z \\
&= (\rho_X \otimes \lambda_Y) \circ (1_X \otimes \varepsilon_Y \otimes \varepsilon_X \otimes 1_Y) \circ \Delta_{X \otimes Y} \circ h \\
&= (\rho_X \otimes \lambda_Y) \circ (1_X \otimes \varepsilon_Y \otimes \varepsilon_X \otimes 1_Y) \circ (1_X \otimes \tau_{X,Y} \otimes 1_Y) \circ (\Delta_X \otimes \Delta_Y) \circ h \\
&= (\rho_X \otimes \lambda_Y) \circ (1_X \otimes \varepsilon_X \otimes \varepsilon_Y \otimes 1_Y) \circ (\Delta_X \otimes \Delta_Y) \circ h \\
&= (\rho_X \otimes \lambda_Y) \circ (\rho_X^{-1} \otimes \lambda_Y^{-1}) \circ h = h.
\end{aligned}$$

Here $\Delta_{X \otimes Y} \circ h = (h \otimes h) \circ \Delta_Z$ since h is a coalgebra morphism.

Hence $m \circ h = m \circ h' \Rightarrow h = h'$. $\square$

The commutative case



Protomodular objects are

- in $BiAlg_{\mathbb{K},coc}$: the Hopf algebras;
- in $BiAlg_{\mathbb{K}}$: none;
- in $BiAlg_{\mathbb{K},c}$: ?

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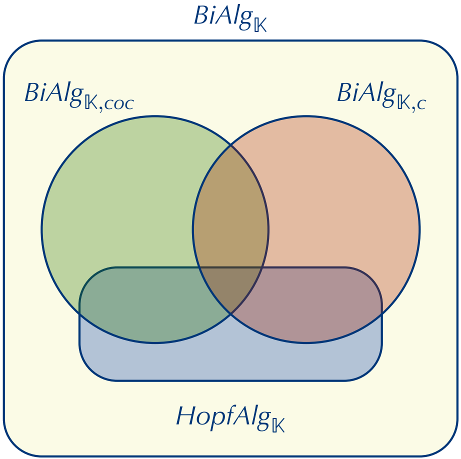

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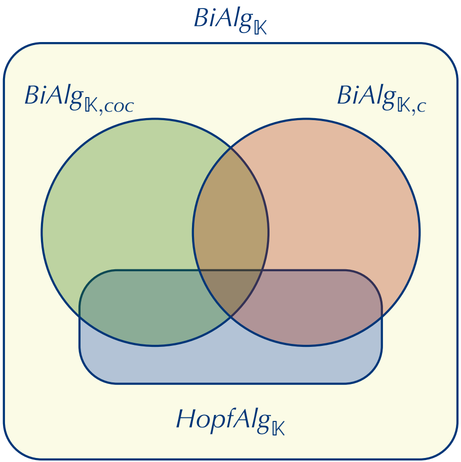

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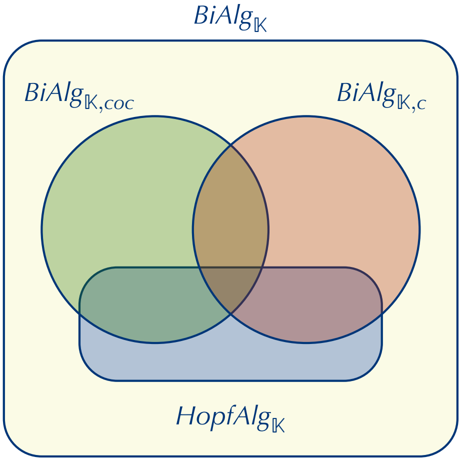

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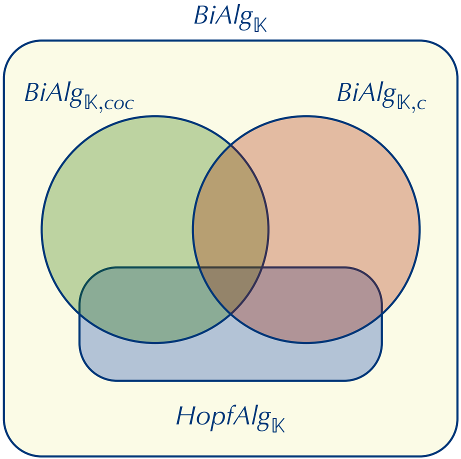

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- commutative Hopf algebras are protomodular in $(BiAlg_{\mathbb{K},c})^{\mathrm{op}}$

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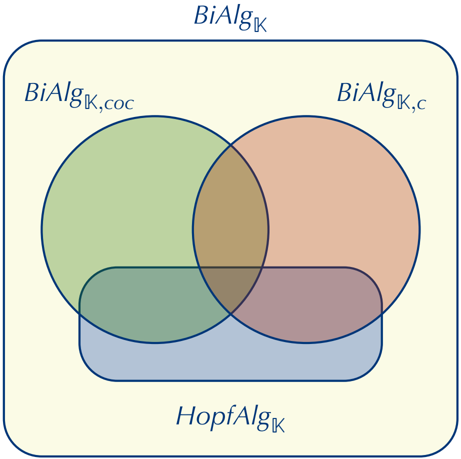

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Via the Yoneda embedding, we see:

- commutative Hopf algebras are protomodular in $(BiAlg_{\mathbb{K},c})^{\mathrm{op}}$
- $HopfAlg_{\mathbb{K},c}$ is **co**protomodular.

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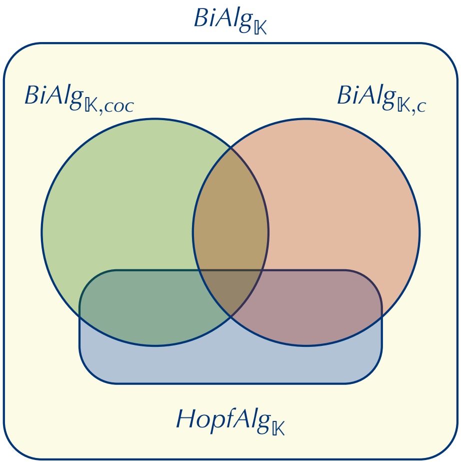

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Via the Yoneda embedding, we see:

- commutative Hopf algebras are protomodular in $(BiAlg_{\mathbb{K},c})^{\mathrm{op}}$
- $HopfAlg_{\mathbb{K},c}$ is coprotomodular.

A consequence of this is that $BiAlg_{\mathbb{K},coc} \cap BiAlg_{\mathbb{K},c} \cap HopfAlg_{\mathbb{K}}$ is an abelian category, as a semi-abelian category which is coprotomodular [Janelidze, Márki & Tholen, 2002]. We regain a result of [Takeuchi, 1972] (and [Grothendieck], in the finite-dimensional case).

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  - What about regularity or Barr-exactness?

Thank you!