Bicategories with lax units and Morita theory

Ülo Reimaa

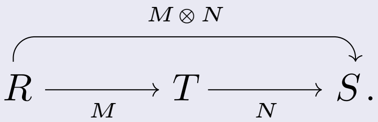
University of Tartu

10.07.2018

For rings with identity and similar structures, several basic properties of the notion of **Morita equivalence** are a consequence of rings and bimodules forming a bicategory.

For rings with identity and similar structures, several basic properties of the notion of **Morita equivalence** are a consequence of rings and bimodules forming a bicategory and a consequence of the notion being essentially the same as the equivalence of objects of that bicategory.

For rings with identity and similar structures, several basic properties of the notion of **Morita equivalence** are a consequence of rings and bimodules forming a bicategory and a consequence of the notion being essentially the same as the equivalence of objects of that bicategory.



For rings with identity and similar structures, several basic properties of the notion of **Morita equivalence** are a consequence of rings and bimodules forming a bicategory and a consequence of the notion being essentially the same as the equivalence of objects of that bicategory.

$$
\begin{array}{c}
\overset{M \otimes N}{\overbrace{\qquad\qquad\qquad\qquad\qquad}} \\
R \xrightarrow[M]{\qquad} T \xrightarrow[N]{\qquad} S\,.
\end{array}
$$

Note the direction of compositional flow.

We have the invertible associator maps

$$a\colon (M \otimes N) \otimes L \to M \otimes (N \otimes L)$$

and the invertible unitors

$$l\colon R \otimes M \to M\,, \quad r \otimes m \mapsto rm\,,$$

$$r\colon M \otimes R \to M\,, \quad r \otimes m \mapsto rm\,.$$

From this point forth, **ring** will mean one that structurally does not have an identity element.

From this point forth, **ring** will mean one that structurally does not have an identity element.

(Unless we specify that we mean a **ring with identity**.)

Rings (without structural identity) and bimodules do not form a bicategory solely because of the fact that the unitors

$$l\colon R \otimes M \to M\,, \quad r \otimes m \mapsto rm\,,$$

$$r\colon M \otimes R \to M\,, \quad m \otimes r \mapsto mr$$

need not be invertible.

Rings (without structural identity) and bimodules do not form a bicategory solely because of the fact that the unitors

$$l\colon R \otimes M \to M\,, \quad r \otimes m \mapsto rm\,,$$

$$r\colon M \otimes R \to M\,, \quad m \otimes r \mapsto mr$$

need not be invertible.

Problem:

How much Morita theory can we still do in a 2-categorical setting?

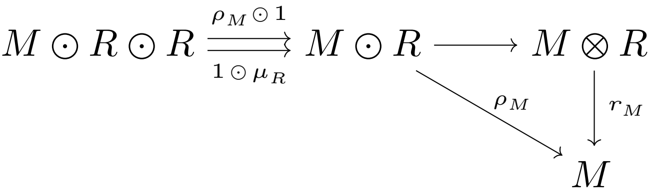
Let R be a ring and let M be a right R-module. In the monoidal category Ab we have the coequalizer

$$M \odot R \odot R \underset{1 \odot \mu_R}{\overset{\rho_M \odot 1}{\rightrightarrows}} M \odot R \longrightarrow M \otimes R$$

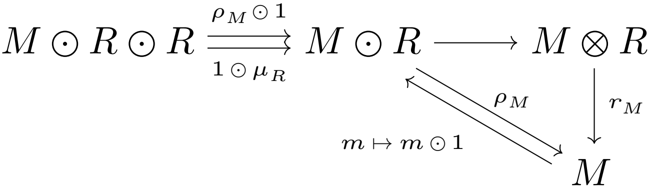
Let R be a ring and let M be a right R-module. In the monoidal category Ab we have the coequalizer

$$
\begin{array}{ccccc}
M \odot R \odot R & \overset{\rho_M \odot 1}{\underset{1 \odot \mu_R}{\rightrightarrows}} & M \odot R & \longrightarrow & M \otimes R \\
 & & & \searrow^{\rho_M} & \\
 & & & & M
\end{array}
$$

Let R be a ring and let M be a right R-module. In the monoidal category Ab we have the coequalizer



Let R be a ring and let M be a right R-module. In the monoidal category Ab we have the coequalizer



Using the map $m \mapsto m \odot 1$ we can show that

$$M \odot R \to M$$

also coequalizes the pair.

The Morita theory of rings without identity and various other internal semigroups objects has been studied by several people and some aspects do fit in a 2-categorical setting.

The Morita theory of rings without identity and various other internal semigroups objects has been studied by several people and some aspects do fit in a 2-categorical setting.

The general themes are the following:

Modules for which the maps

$$l\colon R\otimes M\to M\,,\quad r\otimes m\mapsto rm\,,$$

$$r\colon M\otimes R\to M\,,\quad m\otimes r\mapsto mr$$

are isomorphisms.

The Morita theory of rings without identity and various other internal semigroups objects has been studied by several people and some aspects do fit in a 2-categorical setting.

The general themes are the following:

Modules for which the maps

$$l\colon R \otimes M \to M\,, \quad r \otimes m \mapsto rm\,,$$

$$r\colon M \otimes R \to M\,, \quad m \otimes r \mapsto mr$$

are surjections.

Notation:

- Objects of a bicategory: A, B, ...
- 1-cells of a bicategory: M, N, ...
- 2-cells of a bicategory: f, g, ...
- unit 1-cells: I_A

Lax-unital bicategories

Definition

A **lax-unital bicategory** $\mathcal{B}$ is given by the same data as a bicategory. The properties the data is required to satisfy are also the same, with the following exceptions:

Lax-unital bicategories

Definition

A **lax-unital bicategory** $\mathcal{B}$ is given by the same data as a bicategory. The properties the data is required to satisfy are also the same, with the following exceptions:

- the unitors

  $$l\colon IM \to M\,,$$

  $$r\colon MI \to M$$

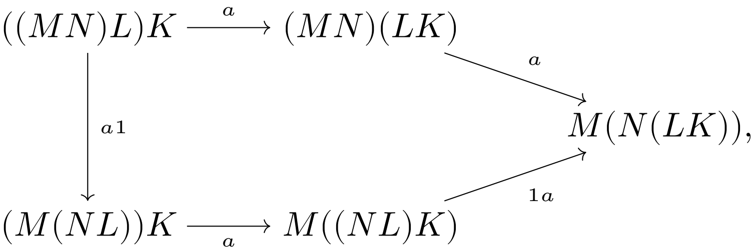
  need not be invertible,

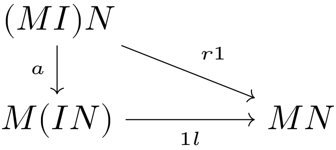
Lax-unital bicategories

Definition

A **lax-unital bicategory** $\mathcal{B}$ is given by the same data as a bicategory. The properties the data is required to satisfy are also the same, with the following exceptions:

- the unitors
  $$l\colon IM \to M\,,$$
  $$r\colon MI \to M$$
  need not be invertible,
- coherence follows from the diagrams below:


$((MN)L)K$
a
$(MN)(LK)$
a
$a1$
$M(N(LK)),$
$(M(NL))K$
a
$M((NL)K)$
$1a$


(MI)N
a
r1
M(IN)
MN
1l

$$
\begin{array}{ccc}
(MI)N & \xrightarrow{\;r1\;} & MN \\
{\scriptstyle a}\downarrow & & \\
M(IN) & \xrightarrow[\;1l\;]{} & MN,
\end{array}
\qquad
\begin{array}{ccc}
(MN)I & \xrightarrow{\;r\;} & MN \\
{\scriptstyle a}\downarrow & & \\
M(NI) & \xrightarrow[\;1r\;]{} & MN,
\end{array}
$$

$$
\begin{array}{ccc}
(IM)N & \xrightarrow{\;l1\;} & MN \\
{\scriptstyle a}\downarrow & & \\
I(MN) & \xrightarrow[\;l\;]{} & MN,
\end{array}
\qquad
II \underset{r}{\overset{l}{\rightrightarrows}} I\,.
$$

Definition

A **Morita context** $\Gamma\colon A \to B$ consists of 1-cells

$$P_\Gamma\colon A \to B \qquad Q_\Gamma\colon B \to A$$

and 2-cells

$$\theta_\Gamma\colon PQ \to I \qquad \phi_\Gamma\colon QP \to I\,.$$

such that the following diagrams commute:

$$\begin{array}{ccc} Q(PQ) & \xrightarrow{1\theta} & QI \\ & & \downarrow r \\ a \uparrow & & Q, \\ & & \uparrow l \\ (QP)Q & \xrightarrow[\phi 1]{} & IQ \end{array} \qquad \begin{array}{ccc} P(QP) & \xrightarrow{1\phi} & PI \\ & & \downarrow r \\ a \uparrow & & P. \\ & & \uparrow l \\ (PQ)P & \xrightarrow[\theta 1]{} & IP \end{array}$$

Lax functors

- natural comparison 2-cells
  $\Phi_{M,N} \colon F(M)F(N) \to F(MN)$,
- comparison 2-cells $\Phi^0_A \colon I_{F(A)} \to F(I_A)$.

$$\begin{array}{ccc} F(M)I_{F(B)} & \xrightarrow{1\Phi^0_B} & F(M)F(I_B) \\ {\scriptstyle r_{F(M)}}\downarrow & & \downarrow{\scriptstyle \Phi_{M,I}} \\ F(M) & \xleftarrow[F(r)]{} & F(MI_B), \end{array}$$

Lax functors

- natural comparison 2-cells
  $\Phi_{M,N} \colon F(M)F(N) \to F(MN)$,
- comparison 2-cells $\Phi^0_A \colon F(I_A) \to I_{F(A)}$.

$$\begin{array}{ccc} F(M)I_{F(B)} & \xleftarrow{1\Phi^0_B} & F(M)F(I_B) \\ {\scriptstyle r_{F(M)}}\downarrow & & \downarrow{\scriptstyle \Phi_{M,I}} \\ F(M) & \xleftarrow[F(r)]{} & F(MI_B), \end{array}$$

Lax functors take Morita contexts to Morita contexts.

$$PQ \xrightarrow{\theta} I$$

$$\downarrow$$

$$F(P)F(Q) \xrightarrow{\Phi} F(PQ) \xrightarrow{\theta} F(I) \xrightarrow{\Phi^0} I\,.$$

From this point forth we will suppose that every hom-category $\mathcal{B}(A, B)$ carries an orthogonal factorization system $(\mathcal{E}, \mathcal{M})$, where

$$\mathcal{E} = \text{strongly epimorphic } 2\text{-cells}$$

and

$$\mathcal{M} = \text{monomorphic } 2\text{-cells.}$$

From this point forth we will suppose that every hom-category $\mathcal{B}(A, B)$ carries an orthogonal factorization system $(\mathcal{E}, \mathcal{M})$, where

$$\mathcal{E} = \text{strongly epimorphic } 2\text{-cells}$$

and

$$\mathcal{M} = \text{monomorphic } 2\text{-cells.}$$

The composition functor of $\mathcal{B}$ is required to map elements of $\mathcal{E}$ into $\mathcal{E}$.

From this point forth we will suppose that every hom-category $\mathcal{B}(A, B)$ carries an orthogonal factorization system $(\mathcal{E}, \mathcal{M})$, where

$$\mathcal{E} = \text{strongly epimorphic } 2\text{-cells}$$

and

$$\mathcal{M} = \text{monomorphic } 2\text{-cells.}$$

The composition functor of $\mathcal{B}$ is required to map elements of $\mathcal{E}$ into $\mathcal{E}$.

$$f \in \mathcal{E} \Rightarrow f1 \in \mathcal{E} \wedge 1f \in \mathcal{E}\,.$$

Definition

We define a 1-cell $M\colon A \to B$ to be

- **right unitary** when $r\colon MI \to M$ lies in $\mathcal{E}$,
- **right firm** when $r\colon MI \to M$ is invertible.

Definition

An object A is called *firm* or *unitary* if the corresponding 1-cell I_A is so.

Definition

Let $\Gamma \colon A \to B$ be a Morita context. When P_Γ and Q_Γ are unitary 1-cells and the 2-cells θ_Γ and ϕ_Γ belong to $\mathcal{E}$, we will call Γ an $\mathcal{E}$-*Morita context*.

Definition

Let $\Gamma\colon A \to B$ be a Morita context. When P_Γ and Q_Γ are unitary 1-cells and the 2-cells θ_Γ and ϕ_Γ belong to $\mathcal{E}$, we will call Γ an *$\mathcal{E}$-Morita context*.

Proposition

The relation of $\mathcal{E}$-equivalence is a transitive and symmetric relation on the objects of a lax-unital bicategory.

Definition

Let $\Gamma \colon A \to B$ be a Morita context. When P_Γ and Q_Γ are unitary 1-cells and the 2-cells θ_Γ and ϕ_Γ belong to $\mathcal{E}$, we will call Γ an $\mathcal{E}$-*Morita context*.

Proposition

The relation of $\mathcal{E}$-equivalence is a transitive and symmetric relation on the objects of a lax-unital bicategory.

Proposition

Let A and B be arbitrary objects of a lax-unital bicategory and suppose that there exists an $\mathcal{E}$-Morita context from A to B. Then A and B are unitary.

Theorem

Suppose that $\Gamma \colon A \to B$ *is a Morita context in a lax-unital bicategory* $\mathcal{B}$*, where either all left unitors or all right unitors are epimorphisms. Then, if* $\theta_\Gamma \colon PQ \to I$ *is in* $\mathcal{E}$*, it is a monomorphism.*

In practice some lax-unital bicategories have a certain absorption property.

In practice some lax-unital bicategories have a certain absorption property.

Absorption

If M is right unitary and N is left unitary, then the 2-cell

$$MIN \to MN$$

is invertible.

In practice some lax-unital bicategories have a certain absorption property.

Absorption

If M is right unitary and N is left unitary, then the 2-cell

$$MIN \to MN$$

is invertible.

Corollary

If $M\colon A \to B$ is right unitary and B is unitary, then $AI\colon A \to B$ is right firm.

In practice some lax-unital bicategories have a certain absorption property.

Absorption

If M is right unitary and N is left unitary, then the 2-cell

$$MIN \to MN$$

is invertible.

Corollary

If $M \colon A \to B$ is right unitary and B is unitary, then $AI \colon A \to B$ is right firm.

Corollary

If A is unitary, then $II \colon A \to A$ is a firm 1-cell.

Using the $(\mathrm{StrongEpis}, \mathrm{Monos})$-factorizations, we can define a lax functor

$$U\colon \mathcal{B}^U \to U\mathcal{B}^U \,.$$

Using the $(\mathrm{StrongEpis}, \mathrm{Monos})$-factorizations, we can define a lax functor

$$U\colon \mathcal{B}^U \to U\mathcal{B}^U .$$

We can turn 1-cells right unitary using the identity on objects lax-functor

$$R\colon \mathcal{B}^U \to U^R\mathcal{B}^U .$$

$$\begin{array}{ccccc}
 & & \overset{r_M}{\frown} & & \\
MI & \xrightarrow[e_M]{} & R(M) & \xrightarrow[m_M]{} & M \\
{\scriptstyle f1}\downarrow & & \downarrow{\scriptstyle R(f)} & & \downarrow{\scriptstyle f} \\
NI & \xrightarrow{e_N} & R(N) & \xrightarrow{m_N} & N . \\
 & & \underset{r_N}{\smile} & &
\end{array}$$

This lax functor is locally right adjoint to the inclusion

$$U^R\mathcal{B}^U \to \mathcal{B}^U\,.$$

If we do the construction of R starting from $\mathcal{B}$, we get a locally well-copointed lax functor

$$R' \colon \mathcal{B} \to \mathcal{B}\,.$$

If the transfinite sequence of 1-cells

$$\cdots \to R'^2(M) \to R'^1(M) \to M$$

always converges, then we get a lax functor

$$R \colon \mathcal{B} \to U^R\mathcal{B}\,.$$

Problem

What if we do the same for the transfinite sequence

$$\cdots \to MII \to MI \to M?$$

Does this converge in the main examples?

$$\cdots \to M \otimes R \otimes R \to M \otimes R \to M\,.$$

Problem

What if we do the same for the transfinite sequence

$$\cdots \to MII \to MI \to M?$$

Does this converge in the main examples?

$$\cdots \to M \otimes R \otimes R \to M \otimes R \to M\,.$$

We can assume that M is unitary.

We want the inclusions

$$U\mathcal{B}^U \to \mathcal{B}^U$$

and

$$F\mathcal{B}^F \to U\mathcal{B}^F$$

to locally have right adjoints, because that allows us to carry any closed structure on $\mathcal{B}$ onto $U\mathcal{B}^U$ and $F\mathcal{B}^F$.

Theorem

Let $\mathcal{B}$ be a right closed lax-unital bicategory in which the 2-cell factorizations in $\mathcal{B}$ are given by the epimorphic and the monomorphic 2-cells. Then, if two firm objects A and B of $\mathcal{B}$ are $\mathcal{E}$-equivalent, the categories $F^R\mathcal{B}(C, A)$ and $F^R\mathcal{B}(C, B)$ are also equivalent for any firm object C of $\mathcal{B}$.