On an algebraic duality and sifted colimits

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(1) **Algebraic Functors**. Given varieties V_i $(i = 1, 2)$, presented via algebraic theories T_i (i.e., V_i is the category **Mod** T_i of all presheaves on T_i preserving finite products) recall that a functor $F : V_1 \to V_2$ is called algebraic iff it is induced by a theory morphism $F_0 : T_2 \to T_1$ (i.e., a functor preserving finite products) in the sense that $F(-) = (-) \cdot F_0$, provided that F_0 is the identity map on objects. The latter is equivalent to F commuting with the forgetful functors $V_i \to \mathbf{Set}$. When viewing varieties as abstract categories, the natural concept of morphism of varieties is an *abstract algebraic functor*: this is a functor $F : V_1 \to V_2$ naturally isomorphic to one induced, in the above sense, by an *arbitrary* theory morphism $F_0 : T_2 \to T_1$.

Theorem: A functor between varieties is abstract algebraic iff it is a right adjoint preserving filtered colimits and regular epimorphisms.

(2) **Sifted colimits**. Recall that filtered colimits are D-colimits for all small categories $\mathbf{D}$ such that, in **Set**, finite limits commute with $\mathbf{D}$-colimits. We introduce sifted colimits as $\mathbf{D}$-colimits for all small categories $\mathbf{D}$ such that, in Set, finite products commute with $\mathbf{D}$-colimits.

Examples: (1) filtered colimits, (2) reflexive coequalizers, (3) $\mathbf{D}$-colimits whenever $\mathbf{D}$ has finite coproducts. A sufficient condition on $\mathbf{D} \neq \emptyset$ (which, luckily, turns out to be necessary too) has been found by C. Lair: for any pair x, y of objects the category of all x-y-cospans in $\mathbf{D}$ be connected; Lair calls such categories tamisante.

Proposition: Abstract algebraic functors are precisely those functors between varieties which preserve limits and sifted colimits.

(3) **Algebraic Duality**. Denote by **VAR** the 2-category of all (finitary, many-sorted) varieties, all abstract algebraic functors, and all natural transformations. Further let **TH** be the 2-category of all Cauchy complete algebraic theories (i.e., small categories with finite products), all theory morphisms, and all natural transformations.

Theorem: **VAR** is dually biequivalent to **TH**.

Corollary: Every variety has an, up to equivalence unique, Cauchy complete algebraic theory.

The proof of the above theorem closely follows the lines of the Gabriel-Ulmer duality: the formation of categories of models represents a 2-functor $Mod : \mathbf{TH}^{op} \to \mathbf{VAR}$ which is proved to be a biequivalence. It is interesting that, although Gabriel

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and Ulmer proved the above corollary, they did not follow the duality line.

References

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[3] C. Lair, *Sur le genre d'esquissibilité des catégories modelables possédant les produits de deux*, Diagrammes 35 (1996) 25-52.