Limits and pointwise colimits in accessible categories

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First consider the following problem: given a class $\mathcal{I}$ of small categories, describe a class $\mathcal{S}$ of sketches such that a category $\mathbf{A}$ is accessible and has $\mathcal{I}$-limits if and only if it is equivalent to $\mathrm{Mod}(\mathbf{S})$ for some sketch $\mathbf{S}$ in S. To quote only but the two easiest examples:

Theorem 1. [Ageron95] *A category* $\mathbf{A}$ *is* α*-accessible and has nonempty limits if and only if it is equivalent to* $\mathrm{Mod}(\mathbf{S})$ *for some sketch* $\mathbf{S}$ *all of whose cones are of size* $< \alpha$ *and all of whose cocones are based on the empty category.*

Theorem 2. [Lair96] *A category* $\mathbf{A}$ *is accessible and has a terminal object if and only if it is equivalent to* $\mathrm{Mod}(\mathbf{S})$ *for some sketch* $\mathbf{S}$ *all of whose cocones are based on connected categories.* (Note: no control of the rank is possible here.)

Other cases have been treated, either by Lair or by myself: accessible categories *with pullbacks and equalizers* [Ageron95], *with pullbacks* [Ageron96], *with equalizers* [Lair97], etc. In each case the problem amounts to determine the class of colimits (*not* meaning the class of indexations) which commute to the considered type of limits in **Set**.

In general, there is no hope to solve the similar problem for colimits: unlike limits, they cannot be *forced* to be pointwise. Accidentally however, there is an analogue to Theorem 1:

Theorem 3. [Ageron97] *A category* $\mathbf{A}$ *is* α*-accessible and has nonempty colimits if and only if it is equivalent to* $\mathrm{Mod}(\mathbf{S})$ *for some sketch* $\mathbf{S}$ *all of whose cones are of size* $< \alpha$ *and all of whose cocones:* (i) *share their vertex with a cone based on the empty category; and* (ii) *are based on categories with non empty limits of size* $< \alpha$ *with the corresponding limit cones being distinguished in* $\mathbf{S}$.

Corollary (of Theorems 1 and 3). *The full subcategories of* α*–ACC consisting respectively of* α*-accessible categories with nonempty limits and of* α*-accessible categories with nonempty colimits are Cartesian closed.* (Note: α–ACC itself is not Cartesian closed.)

Any analogue to Theorem 2 has to be of a different form, involving a generalised notion of accessibility. First, say that an object A in a category with an initial object 0 is **nonempty** if $\mathrm{Hom}(A, 0) = \emptyset$. Say that a category $\mathbf{A}$ is α**-positively accessible** if $\mathbf{A}$ has α-filtered colimits, an initial object, and a small full subcategory $\mathbf{B}$ consisting of nonempty α-presentable objects so that every object of $\mathbf{A}$ is an empty or α-filtered colimit of objects in $\mathbf{B}$. Then:

Theorem 4. [Ageron99] *A category* $\mathbf{A}$ *is positively accessible if and only if it is* *equivalent to* Mod($\mathbf{S}$) *for some sketch* $\mathbf{S}$ *all of whose cones are based on nonempty categories.*

Now let $\mathbf{S}$ be a sketch. Define $\mathbf{S}^+$ as $\mathbf{S}$ with a terminal object S added and the following items distinguished: the cone (id_S, id_S) based on the discrete category with two objects, the cones of $\mathbf{S}$ completed with the arrows from them to S, the same cocones as those of $\mathbf{S}$.

Theorem 5. [Ageron99] (i) *Let* $\mathbf{S}$ *be a sketch all of whose cones are based on nonempty* α*-small categories. If* Mod($\mathbf{S}$) *is* α*-positively accessible, then* Mod($\mathbf{S}$) *is* α*-accessible.*
(ii) Let $\mathbf{S}$ *be a sketch all of whose cones are of size* $< \alpha$*. If* Mod($\mathbf{S}$) *is* α*-accessible, then* Mod($\mathbf{S}^+$) *is* α*-positively accessible.*

Results similar to Theorems 4 and 5 apply for example to sketches all of whose cones are based on *connected* categories: in this case Theorem 5 gives a result announced by S. Lack.