Infinitary linear combinations in cotorsion modules

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Let R be an integral domain which is not a field. Moreover, assume that R is complete and first-countable in the R-topology, i.e. in the unique ring topology in which the non-zero ideals form a basis of 0-neighbourhoods. An R-module M is called cotorsion (in the sense of Matlis) if $\mathrm{Hom}_R(K/R, M) = 0$ and $\mathrm{Ext}^1_R(K/R, M) = 0$, where K is the quotient field of R. Then it is more or less well-known that the cotorsion modules form a full reflective subcategory $\mathcal{C}$ of the category of all R-modules. We observed that $\mathcal{C}$ can be described as the category of all R-modules M admitting infinitary linear combinations $\sum_{n\in\mathbb{N}} \alpha_n x_n$ where $x_n \in M$ for all $n \in \mathbb{N}$, and $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in R which converges to 0 in the R-topology, such that $\sum_{n\in\mathbb{N}} \alpha_n(\sum_{m\in\mathbb{N}} \beta_{nm} x_m) = \sum_{m\in\mathbb{N}}(\sum_{n\in\mathbb{N}} \alpha_n \beta_{nm}) x_n$ holds whenever the left-hand side is defined. In particular, $\mathcal{C}$ is a variety of rank $\aleph_1$. The above infinitary linear combinations are automatically unique, and they are preserved by all module homomorphisms between cotorsion modules. Moreover, $\mathcal{C}$ is abelian and (obviously) locally countable presentable, but not locally finitely presentable. We show some more properties of $\mathcal{C}$, e.g. the non-existence of a cogenerator.

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