Bernays-Gödel type theory

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One slogan of categorical logic is that categories behave like categories of sets. This is backed up by results such that regular categories support exactly regular logic (the $\exists$-$\wedge$ fragment of intuitionistic first-order logic), or that Heyting categories support exactly first-order intuitionistic logic, just to name two examples. One should note that along the correspondence between classes of categories and logics we get on the logical side various types of type theories. This remark becomes even more important when we look at elementary toposes and intuitionistic higher-order logic, and recall that elementary toposes (with a natural numbers object) provide a good universe in which one can do mathematics. Thus they may be seen as an alternative to the classical foundation of mathematics based on Zermelo-Fraenkel set-theory.

In this talk I will take the position that categories are categories of *classes*. Accordingly, we want to axiomatize what it means to be a *small* class, that is, a set: we are looking for the categorical analogue of Bernays-Gödel set-theory.

The starting point will be a regular category. *Small structure* on it is for every object X and every index-object I a collection of I-indexed families of subobjects of X, referred to as families of small subobjects of X. The axioms we consider are (i) axioms that ensure that the notion of a family of small subobjects of X depends only on X, and not on the indexing object; (ii) axioms very close to the standard formulation of Bernays-Gödel set-theory; (iii) and the axiom of *representability*, which says that for each object X the class $P_S(X)$ of all small subobjects of X exists in the underlying category.

The first surprise is that the axioms above force the underlying regular category to be a Heyting category with a subobject classifier and a natural numbers objects (in a slightly different setting this result was independently observed by Alex Simpson). During the talk I will show how the axioms of Bernays-Gödel set-theory can be reformulated using the internal logic of the underlying category. For example, axiom (Replacement) makes P_S into a covariant functor which is part of a monad. The unit are the singleton maps $X \to P_S(X)$, and multiplication is given by union $P_S(P_S(X)) \to P_S(X)$. After this (elementary) reformulation it will be apparent that every object X such that $X \cong P_S(X)$, thus for example an initial algebra for the powerset monad described above, is a model of intuitionistic set-theory.

Finally we discuss the close relationship between our research and the algebraic set-theory of André Joyal and Ieke Moerdijk.