

# Some points on Grothendieck's interpretation of Galois Theory

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We analyze Grothendieck's 1960-61 section 4 of the expose V in SGA 1 where he states some axioms on a category which characterize the category of continuous actions on finite sets of a profinite group. In doing so he introduces an interpretation of classical Galois theory which leads to categorical developments much beyond the original Galois's scope.

0) We show a characterization of (the category of) all transitive actions of a discrete group. This corresponds exactly to Galois's Galois theory, and the arguments are exactly those utilized in the theory of the fundamental group of a topological space in the case in which there exists a universal covering. This corresponds to the case where the conservative fiber functor is representable and all the objects are connected.

There are two generalizations in very different directions:

1) A straightforward one, namely, that of all (non necessarily connected) objects and a small conservative set of representable fiber functors (inverse images of points). This is a characterization of presheaf categories attributed by Grothendieck (in the SGA 4 1963-4) to J.E. Roos. The additive version of this was developed by B. Mitchell and P. Freyd also in the earlier sixties. If we do not require the representable functors to be fiber functors, we are led naturally to Giraud's Theorem of characterization of topoi.

2) Grothendieck's fundamental theorem, where it is not required for the fiber functor to be representable. Here it appears the profinite group, and the arguments are those of the Galois theory of all finite extensions inside the algebraic closure. We show how this theorem is obtained by passing to the limit in a filtered colimit of equivalences given by 0). It is straightforward to pass then to all (not necessarily connected) objects. This corresponds to a characterization of connected locally connected boolean topoi (there is work on this by M.Barr-R Diaconescu and P. Johnstone). We show how this characterization is obtained by passing to the limit in a filtered inverse limit of topoi and geometric morphisms. Before the limit, the groups are finite, which leaves out the case of all covering spaces when the universal covering does not exist. However, there is a theorem (generalizing Grothendieck's and proven by the same methods) that covers this case.

3) After all this it comes Joyal-Tierney theory.