Global dimension of the diagram category on a totally ordered set

Ahmet A. Husainov*

Any partially ordered set $(J, \leq)$ will be considered as a small category J such that $ObJ = J$ and $MorJ$ consists of all pairs $x \leq y$ in J. A subset $U \subseteq J$ is *open* if $u \in U$ & $u < x \in J$ implies $x \in U$. A subset $I \subseteq J$ is *coinitial* if for every $x \in J$ there exists $y \in I$ such that $y \leq x$. The *coinitiality* of J is the inf of the cardinalities of coinitial subsets.

Let K be a ring with identity. Denote by Mod_K^J the category of all functors from J to the category Mod_K of left K–modules. For any abelian category $\mathcal{A}$ we denote by $gl.dim\mathcal{A}$ the global dimension in the sense of [1].

Theorem. *Let J be a totally ordered set which contains at least two elements, K be a commutative Noetherian ring with identity. If K is a Dedekind domain or a local ring then*

$$gl.dimMod_K^J = n + 2 + gl.dimMod_K,$$

where $\aleph_n$ is the sup of the coinitialities of open subsets $U \subset J$ such that $U \neq J$.

Mitchell's generalization [1, corollary 36.12] of the Osofsky statement [2, corollary 7.5] to the valuation categories gives equality

$$gl.dimMod_K^J = n + 2$$

for each division ring K. Brune [3] proved the inequality

$$gl.dimMod_K^J \leq n + 2 + gl.dimMod_K$$

for every commutative Noetherian ring K.

Let $\mathbf{Z}$ be the ring of integers. Considering $K = \mathbf{Z}$ in the theorem we get author's result [4]. For the ordered set $\mathbf{R}$ of reals we get $gl.dimMod_{\mathbf{Z}}^{\mathbf{R}} = 3$. This is the answer to the question of Brune [3].

References

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