

The theory of higher category in a few steps

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There are many obstacles to the development of a theory of higher categories. One is the algebraic complexity of higher categories and the other is the absence of a clear picture of what these objects should be. The goal of our work is to contribute to the solution of these problems. It has become an accepted idea that higher categories should model homotopy types in the form of higher groupoids. We thus have a criterion for checking the correctness of a definition of higher categories but it only applies to a restricted class, namely to higher groupoids. This limitation is a consequence of the fact that classical homotopy theory is based on reversible paths. There are many natural examples in physics and mathematics of spaces with irreversible paths: future can be distinguished from the past; potential energy is giving a direction to a mechanical system; a morse function is defining a direction on a manifold; a scheme has a natural partial ordering; a Grothendieck topos has Sierpinsky paths; a simplicial sets has irreversible arrows. More general examples of higher categories should thus be obtained by studying these spaces. In the present work we shall restrict to certain cellular sets extending simplicial sets. We shall define higher categories by horn filling conditions. In the case of simplicial sets they are the restricted Kan complexes in the sense of Boardman and Vogt (we are calling them V -categories). They are the fibrant objects for a certain Quillen model structure on simplicial sets. On the basis of our result we claim that most if not all of ordinary category theory can be extended to V -categories. We then introduce a concept of higher disks, an interval been a 1-disk. An $(n+1)$ -disks is an n -disk with fixed boundary moving. The classifying topos for disks is a category of cellular sets having all the good properties of the category of simplicial sets including geometric realisation. There is a concept of horns and of inner horns. By using these horns we obtain a model structure on cellular sets and higher categories are defined to be the fibrant objects. The category of (weak) n -categories is cartesian closed. We show that any $(n+1)$ -category is weakly equivalent to a category enriched over n -categories. We end up by comparing our work with other approaches to higher categories. The closest in philosophy is by Hirschowitz-Simpson-Tansamani. The operadic approach of Batanin is fitting well in the picture.