# Tannaka duality for Maschkean categories

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This seminar aims to generalize classical Tannaka duality by replacing the category of vector spaces with more general monoidal categories.

Recall that given a coalgebra $C$ in the category of vector spaces, one may form the category $\mathrm{Comod}_f(C)$ of finite dimensional representations of $C$ and there is the forgetful functor into the category $\mathrm{Vect}_f$ of finite dimensional vector spaces. Conversely, given a category $\mathcal{C}$ equipped with a functor $X : \mathcal{C} \to \mathrm{Vect}_f$, one may form the coalgebra $\mathrm{End}^\vee(X)$ in the category of vector spaces. A fundamental result [1] of Tannaka duality is that when $X$ is the forgetful functor $\mathrm{Comod}_f(C) \to \mathrm{Vect}_f$, the coalgebra $\mathrm{End}^\vee(X)$ is canonically isomorphic to $C$. Tannaka duality also shows that there is a correspondence between monoidal structures on $\mathrm{Comod}_f(C)$ with the forgetful functor being strong monoidal, and bialgebra structures on $C$. Similarly, there is a correspondence between braidings on $\mathrm{Comod}_f(C)$ and cobraidings on $C$.

This seminar aims to generalize this theory by replacing coalgebras in the category of vector spaces with comonoids in a general braided monoidal category $\mathcal{V}$. First, a 2-functor

$$\mathrm{Comod}_f : \mathrm{Comon}(\mathcal{V}) \to \mathcal{V}_f\text{-}\mathrm{Act}/\mathcal{V}_f$$

is defined whose value on a comonoid in $\mathcal{V}$ is its category of representations. When $\mathcal{V}$ is a cocomplete monoidal category, this 2-functor has a left biadjoint $E$. The unit $\eta : C \to E(\mathrm{Comod}_f(C))$ of this biadjunction is not in general an equivalence, although it is so when $\mathcal{V}$ is the monoidal category of vector spaces. The problem of reconstructing a comonoid from its representations may be stated as finding sufficient conditions on $\mathcal{V}$ that ensure that this unit is an equivalence. A class of braided monoidal categories, called *Maschkean categories*, is defined and it is shown that for these categories the unit is an isomorphism. It is further shown that when $\mathcal{V}$ is a *symmetric* Maschkean category, $\mathrm{Comod}_f : \mathrm{Comon}(\mathcal{V}) \to \mathcal{V}_f\text{-}\mathrm{Act}/\mathcal{V}_f$ is *monoidally bi-fully-faithful*, and this makes precise the idea that there is a correspondence between monoidal structures (respectively braided monoidal structures) on $\mathrm{Comod}_f(C)$ and coquasi-bialgebra (respectively cobraided coquasi-bialgebra) structures on $C$. Of course, the category of vector spaces is a symmetric Maschkean category and it is shown that if $\mathcal{V}$ is a symmetric Maschkean category and $H$ is a cosemisimple cobraided Hopf algebra in $\mathcal{V}$ then $\mathrm{Comod}(H)$ is again a Maschkean category. This allows us to conclude that many categories, including graded vector spaces, representations of a finite group, representations of a compact group, and representation of a compact quantum group, are Maschkean categories.

## References

[1] A. Joyal and R. Street, *An introduction to Tannaka duality and quantum groups*, Springer Lecture Notes in Mathematics 1488 (1991), 411-492.