Duality between homomorphisms and state mappings of concrete dynamic algebras

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Given a set S, let $exp(S)$ be the Boolean algebra of all the subsets of S endowed with the set-theoretical operations of union, intersection, complementation and the empty set. A ***Boolean Kripke structure on S*** is a 3-tuple $(\mathcal{B}, \mathcal{A}, S)$, where $\mathcal{A}$ is a family of binary relations on S closed under union, composition and reflexive transitive closure and $\mathcal{B}$ is a Boolean subalgebra of $exp(S)$ which contains the pre-image $R^{-1}(G)$ of every $G \in \mathcal{B}$ under every $R \in \mathcal{A}$. Therefore, Boolean Kripke structures are two-sorted universal algebras of the type $(2, 2, 1, 0)$ on the Boolean sort, $(2, 2, 1)$ on the so called regular sort and one heterogeneous operation $<>: \mathcal{A} \times \mathcal{B} \to \mathcal{B}$. Boolean Kripke structures, which can model the input-output behaviour of computer programs, are the traditional models of Propositional Dynamic Logic.

The dynamic algebras form the smallest variety in the class of all two-sorted universal algebras of the type described above such that (i) this variety contains all Boolean Kripke structures and (ii) it is determined by a set of equations in the Boolean sort (cf. [3] to see the set of equations). This property is the algebraic counterpart of the Completeness Theorem for Propositional Dynamic Logic.

We say that a dynamic algebra is ***concrete*** if it is isomorphic to a Boolean Kripke structure.

A ***state mapping*** $f : (\mathcal{B}, \mathcal{A}, S) \longleftarrow (\mathcal{B}', \mathcal{A}', S')$ is an arbitrary mapping $f : S' \to S$ satisfying $f^{-1}(G) \in \mathcal{B}'$ for every $G \in \mathcal{B}$.

We verify that, in general, there is no relationship between homomorphisms and state mappings. Nevertheless, we have

Theorem. *There is a class of separable concrete dynamic algebras on which the category of homomorphisms is dual to the category of state mappings. The category is so large that it contains every category of universal algebras as a full subcategory.*

Corollary. *For every monoid M there exists a concrete dynamic algebra with the endomorphism monoid isomorphic to M and the monoid of state mappings isomorphic to the opposite monoid.*

This is of particular interest, since there is a group which cannot be represented as the automorphism group of any Boolean algebra (cf. [1]).

Then, given a monoid M and one concrete dynamic algebra $\mathcal{D}_M$ which satisfies $M \cong End(\mathcal{D}_M)$, we put the problem of characterizing the submonoid of $End(\mathcal{D}_M)$ which represents M'. For that purpose, we will introduce the concept of ***lattice***

Kripke structure.

References

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