Equational hull of varieties

Jiří Rosický*

Our aim is to understand which categories satisfy the same "equational properties" as varieties. It is explained in the abstract of J. Adámek that the right morphisms among varieties are functors preserving limits and sifted colimits. In this way, we obtain the 2-category VAR of varieties and we want to describe its equational hull over CAT. We are able to do it with respect to "small" operations which are given by limits and sifted colimits. The resulting hull consists of categories $\mathcal{K}$ having limits and sifted colimits such that the colimit functor $Sind\,\mathcal{K} \to \mathcal{K}$ from a free completion $Sind\,\mathcal{K}$ of $\mathcal{K}$ under sifted colimits preserves limits. We call such categories $\mathcal{K}$ *completely exact*. In fact, the 2-monad Lim given by a free completion under limits lifts to the 2-category of $Sind$-pseudoalgebras. Completely exact categories are then precisely pseudoalgebras for the lifted 2-monad. Any completely exact category $\mathcal{K}$ is exact, any product of regular epimorphisms in $\mathcal{K}$ is a regular epimorphism and filtered colimits in $\mathcal{K}$ commute with finite limits and distribute over all products. There are completely exact categories which are not varieties. For example, any essential localization of a variety is completely exact. Essential localizations of varieties are well understood in the additive context – they are precisely Grothendieck categories satisfying AB4* and AB6.

We have not been able to describe the equational hull of VAR over CAT w.r.t. all operations but we know that it contains any continuous lattice. Hence VAR is not pseudomonadic over CAT.

*Joint work with J. Adámek and F. W. Lawvere.