Some remarks on topological functors

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A theory of topological functors is presented in the book [1]. A topological functor is a faithful functor $U : \mathbf{A} \to \mathbf{X}$ that has unique U-initial lifts of U-structured sources. A theorem that R.-E. Hoffmann proved in his thesis [2]plays a very important role in our investigations. It says that from each category $\mathbf{A}$ there is at most one topological functor into the category **Set** of sets modulo natural isomorphy. In [3] he proves moreover that from each category $\mathbf{A}$ there is at most one topological functor into a balanced category (i.e. a category in which each bimorphism is isomorphic) modulo equivalence. Functors $F : \mathbf{A} \to \mathbf{X}$ and $G : \mathbf{A} \to \mathbf{Y}$ are equivalent iff there exists an equivalence $H : \mathbf{X} \to \mathbf{Y}$ such that $H \circ F \cong G$.

We will see for functors $\mathbf{A} \xrightarrow{U} \mathbf{B} \xrightarrow{V} \mathbf{C}$ if the composition VU and U are topological then so is V. We get that from each category $\mathbf{A}$ there is at most one topological functor into a balanced basecategory modulo isomorphy, i.e. for topological functors $F : \mathbf{A} \to \mathbf{X}$ and $G : \mathbf{A} \to \mathbf{Y}$ into balanced categories there exists an isofunctor $H : \mathbf{X} \to \mathbf{Y}$ such that $H \circ F \cong G$. We will describe the basecategories $\mathbf{X}$ into which there exists at most one topological functor from each category modulo natural isomorphy.

By looking on the above uniqueness-theorems one could expect that in the case of a full subcategory $\mathbf{B} \subseteq \mathbf{A}$ and a topological functor $U : \mathbf{A} \to \mathbf{Set}$ the only chance for a functor from $\mathbf{B}$ to **Set** to be topological is the restriction of U. In general this is not true. But it holds if $(\mathbf{A}, U)$ is a c-construct, i.e. every constant map between (the underlying sets of) objects is a morphism.

References

[1] J. Adámek, H. Herrlich and G. E. Strecker, *Abstract and Concrete Categories*, Wiley-Interscience (1990).

[2] R.-E. Hoffmann, *Die kategorielle Auffassung der Initial- und Finaltopologie*, Bochum, thesis (1974).

[3] R.-E. Hoffmann, *Topological Functors and Factorizations*, Arch. Math. (Basel) **26** (1975), 1-7.