Categorical semantics of non-commutative logic

Paul Ruet*

Non-commutative logic (NL) is a unification of:

– commutative linear logic [5] and

– cyclic linear logic [6, 12], a classical conservative extension of the Lambek calculus [8].

Previous work [9, 1, 10] was devoted to proof nets and sequent calculus for NL. Here we study the categorical semantics of NL. One can say that proof nets constitute already a semantics of proofs of NL, but on one hand the extension to non-multiplicative connectives is delicate (boxes) and on the other hand one would like to have a semantics that is more general than syntax (correctness) and close enough (full completeness)

For non-commutativity, one thinks naturally about Hopf algebras [7]. Indeed the category $\mathrm{Mod}(H)$ of modules over a given Hopf algebra H is monoidal closed, and we know since the works of Barr and Seely [11] notably, that this is the right framework in order to find models of linear logic. In fact, models of LL are $*$-autonomous categories [2] but one can:

–restrict oneself to finite-dimensional modules: $\mathrm{Mod}(H)$is then compact close, or

– use an idea of Lefschetz in the infinite-dimensional case: enrich the modules with a linear topology, and then the sub-category $\mathrm{TMod}(H)$ of $\mathrm{Mod}(H)$ is $*$-autonomous but not compact.

The essential point is that in general $\mathrm{Mod}(H)$ is not symmetric: indeed the monoidal product is determined by the coalgebra structure of H, which need not be symmetric. Varying the Hopf algebra enables then a control on the symmetry / asymmetry of the category at hand. For instance by using Hopf algebras, Rick Blute and Phil Scott have exhibited complete models of LL [3] and cyLL [4].

The following provides a definition of categorical models of the multiplicative fragment MNL of non-commutative logic, that particularize $*$-autonomous categories.

Definition. An *entropic* category is a category C with the following structure:

– $(C, \otimes, 1, \multimap, \bot)$ is a symmetric $*$-autonomous category,

– $(C, \odot, 1, -\!\bullet, \bullet\!-, \bot)$ is a cyclic $*$-autonomous category,

– a monoidal natural transformation $E : \otimes \dot{\to} \odot$ such that the negations coincide.

*Joint work with Rick Blute.

Theorem (Soundness and invariance by reduction). Entropic categories are models of MNL: to a proof π of A is associated a morphism $|\pi| : 1 \to A$ of any entropic category C, and if π reduces to π' then $|\pi| = |\pi'|$.

Examples of entropic categories (therefore models of MNL) are obtained by considering sets G having 2 group structures with same inverse and same unit: Then $K[G]$ has 2 Hopf algebra structures with same multiplication (the diagonal), and one dualizes to get 2 comultiplications (thus 2 tensors) with a single multiplication.

Theorem. Let k be a vector space, p a non zero natural number, n an odd divisor of the Euler indicator $\phi(p)$ and H the Hopf dual of $k[Z_n \times Z_p]$. The reflexive objects in the category TMod(H) and associated morphisms form an entropic category (where one of the 2 tensors is non-commutative).

This result relies on:

– a lemma which states that $Z_n \times Z_p$ indeed has several group structures with same inverse and same unit (semi-direct products),

– the definition of a subset, Core(G), of $G = Z_n \times Z_p$, which induces the definition of the natural transformation E.

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