Objective number theory

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A simple property of finite coproducts shared by categories of space, e.g. any topos or any of the usual categories of algebraic spaces, topological spaces or smooth manifolds, has surprisingly powerful consequences. A category $\mathbf{E}$ with finite coproducts is called 'extensive' if for each pair (a, b) of objects, the obvious functor from the product of the categories $\mathbf{E}/a$ and $\mathbf{E}/b$ to $\mathbf{E}/(a+b)$ is an equivalence, so that a map into a sum is the same as a splitting of its domain with a pair of maps into the pieces. If $\mathbf{E}$ is extensive, then so are $\mathbf{E}/a$ and $\mathbf{E}$-valued presheaves on any small category. If $\mathbf{E}$ has also finite coproducts, the extensive property implies the distributive law, and Lawvere and I have taken to calling the study of addition and multiplication (and exponentiation, if $\mathbf{E}$ is cartesian closed) of isomorphism classes of objects in such categories 'objective number theory', generalizing the case $\mathbf{E} =$ finite sets which gave rise to classical number theory, the arithmetic of natural numbers.

With a mild finiteness restriction, the arithmetic of algebraic objects is indeed very close to that of natural numbers. Without finiteness, the generic extensive category with an object satisfying any given 'fixed point equation' (of importance in the theory of data types and elsewhere) has its number theory reducible by results of Blass and Gates to pure algebra, about 'rigs', or 'commutative rings without negatives', for which we have general techniques demonstrating e.g. that equality of Euler characteristic and of dimension jointly imply isomorphism. We have as yet only a few general results on exponential arithmetic (for cartesian closed extensive categories), but for directed graphs there is an analog of a special case of the speaker's old conjecture on algebraic independence of exponentials, so that we know more about algebraic powers of rational graphs than is yet known about algebraic powers of rational numbers!