A remark on conservative cocompletions of categories

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Given a class $\mathbf{F}$ of small categories, M. Kelly has constructed in [Ke82], for every small category X, what we call a *free* $\mathbf{F}$*-conservative cocompletion*: this is a full embedding $E : \mathsf{X} \longrightarrow \mathsf{X}^*$ with X^* cocomplete, which is $\mathbf{F}$-cocontinuous (i.e., preserves all existing $\mathbf{F}$-colimits in X) and such that any other $\mathbf{F}$-cocontinuous functor H from X to a cocomplete category Y has an essentially unique extension to $H^* : \mathsf{X}^* \longrightarrow \mathsf{Y}$ which preserves all small colimits.

The construction of a free $\mathbf{F}$-conservative cocompletion can be performed for not neccessarily small categories X as well, the resulting X^*, however, need not be a category in the same universe as X.

We show a simple example of a locally small category X such that a free $\mathbf{F}$-conservative cocompletion X^* is not locally small for $\mathbf{F}$ = coequalizers.

We also derive a sufficient condition on a locally small category X such that its free $\mathbf{F}$-conservative cocompletion X^* is locally small:

> For each X-object X there should be at most a small set of vertices C_i, $i \in I$, of non-absolute $\mathbf{F}$-colimit cocones such that $\mathsf{X}(X, C_i)$ is nonempty.

Our condition, in case when $\mathbf{F} = \emptyset$, covers the well-known fact that each locally small category has a locally small free cocompletion.

Whether there exists a usable necessary and sufficient condition for local smallness of a free $\mathbf{F}$-conservative cocompletion of a locally small category is an open problem.

References

[Ke82] G.M. Kelly, *Basic Concepts of Enriched Category Theory*, London Math. Soc. Lecture Notes Series 64, Cambridge Univ. Press, 1982.

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