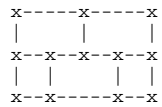


Problem 1 Given 12 vertices and 16 edges arranged as follows:



Draw any curve which does not pass through any vertex. Prove that the curve cannot intersect each edge just once. Intersection means that the curve crosses the edge from one side to the other. For example, a circle which had one of the edges as tangent would not intersect that edge.

Problem 2 Given a rectangle ABCD with AC length e and four circles centers A, B, C, D and radii a, b, c, d respectively, satisfying $a+c=b+d<e$. Prove you can inscribe a circle inside the quadrilateral whose sides are the two outer common tangents to the circles center A and C, and the two outer common tangents to the circles center B and D.

Problem 3 Prove that any 39 successive natural numbers include at least one whose digit sum is divisible by 11.

Problem 4

- (a) Arrange 7 stars in the 16 places of a 4 x 4 array, so that no 2 rows and 2 columns contain all the stars.
- (b) Prove this is not possible for <7 stars.

Problem 5

- (a) Given a quadruple (a, b, c, d) of positive reals, transform to the new quadruple (ab, bc, cd, da). Repeat arbitrarily many times. Prove that you can never return to the original quadruple unless $a=b=c=d=1$.
- (b) Given n a power of 2, and an n-tuple (a_1, a_2, \dots, a_n) transform to a new n-tuple $(a_1a_2, a_2a_3, \dots, a_{n-1}a_n, a_na_1)$. If all the members of the original n-tuple are 1 or -1, prove that with sufficiently many repetitions you obtain all 1s.

Problem 6

- (a) A and B move clockwise with equal angular speed along circles center P and Q respectively. C moves continuously so that $AB=BC=CA$. Establish C's locus and speed.
- * (b) ABC is an equilateral triangle and P satisfies $AP=2, BP=3$. Establish the maximum possible value of CP.

Problem 7 Given an $m \times n$ array of real numbers. You may change the sign of all numbers in a row or of all numbers in a column. Prove that by repeated changes you can obtain an array with all row and column sums non-negative.

Problem 8 Given $n < 1$ points, some pairs joined by an edge (an edge never joins a point to itself). Given any two distinct points you can reach one from the other in just one way by moving along edges. Prove that there are $n-1$ edges.

Problem 9 Given any natural numbers m, n and k. Prove that we can always find relatively prime natural numbers r and s such that $rm+sn$ is a multiple of k.

Problem 10 A and B play the following game with N counters. A divides the counters into 2 piles, each with at least 2 counters. Then B divides each pile into 2 piles, each with at least one counter. B then takes 2 piles according to a rule which both of them know, and A takes

the remaining 2 piles. Both A and B make their choices in order to end up with as many counters as possible. There are 3 possibilities for the rule:

R1 B takes the biggest heap (or one of them if there is more than one) and the smallest heap (or one of them if there is more than one).

R2 B takes the two middling heaps (the two heaps that A would take under R1).

R3 B has the choice of taking either the biggest and smallest, or the two middling heaps. For each rule, how many counters will A get if both players play optimally?

Problem 11 Given three arbitrary infinite sequences of natural numbers, prove that we can find unequal natural numbers m, n such that for each sequence the mth member is not less than the nth member.

Problem 12 120 unit squares are arbitrarily arranged in a 20 x 25 rectangle (both position and orientation is arbitrary). Prove that it is always possible to place a circle of unit diameter inside the rectangle without intersecting any of the squares.

Problem 13 ABCD is any convex quadrilateral. Construct a new quadrilateral as follows. Take A' so that A is the midpoint of DA'; similarly, B' so that B is the midpoint of AB'; C' so that C is the midpoint of BC'; and D' so that D is the midpoint of CD'. Show that the area of A'B'C'D' is five times the area of ABCD.

Problem 14 Given a fixed circle C and a line L through the center O of C. Take a variable point P on L and let K be the circle center P through O. Let T be the point where a common tangent to C and K meets K. What is the locus of T?

Problem 15 Given integers a_0, a_1, \dots, a_{100} , satisfying $a_1 > a_0, a_1 > 0$, and $a_{r+2} = 3a_{r+1} - 2a_r$ for $r=0, 1, \dots, 98$. Prove $a_{100} > 2^{99}$.

Problem 16 Prove that there are no integers a, b, c, d such that the polynomial ax^3+bx^2+cx+d equals 1 at $x=19$ and 2 at $x=62$.

Problem 17 Given an $n \times n$ array of numbers. n is odd and each number in the array is 1 or -1. Prove that the number of rows and columns containing an odd number of -1s cannot total n.

Problem 18 Given the lengths AB and BC and the fact that the medians to those two sides are perpendicular, construct the triangle ABC.

Problem 19 Given four positive real numbers a, b, c, d such that $abcd=1$, prove that $a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd \geq 10$.

Problem 20 Given a fixed regular pentagon ABCDE with side 1. Let M be an arbitrary point inside or on it. Let the distance from M to the closest vertex be r_1 , to the next closest be r_2 and so on, so that the distances from M to the five vertices satisfy $r_1 \leq r_2 \leq r_3 \leq r_4 \leq r_5$. Find (a) the locus of M which gives r_3 the minimum possible value, and (b) the locus of M which gives r_3 the maximum possible value.

Problem 21 Given a number with 1998 digits which is divisible by 9. Let x be the sum of its digits, let y be the sum of the digits of x, and z the sum of the digits of y. Find z.

Problem 22 $AB=BC$ and M is the midpoint of AC . H is chosen on BC so that MH is perpendicular to BC . P is the midpoint of MH . Prove that AH is perpendicular to BP .

Problem 23 The triangle ABC satisfies $0 \leq AB \leq 1 \leq BC \leq 2 \leq CA \leq 3$. What is the maximum area it can have?

Problem 24 Given unequal integers x, y, z prove that $(x-y)^5 + (y-z)^5 + (z-x)^5$ is divisible by $5(x-y)(y-z)(z-x)$.

Problem 25 Given a_0, a_1, \dots, a_n , satisfying $a_0 = a_n = 0$, and $a_{k-1} - 2a_k + a_{k+1} \geq 0$ for $k=0, 1, \dots, n-1$. Prove that all the numbers are negative or zero.

Problem 26 Given two sets of positive numbers with the same sum. The first set has m numbers and the second n . Prove that you can find a set of less than $m+n$ positive numbers which can be arranged to part fill an $m \times n$ array, so that the row and column sums are the two given sets.

Example: row sums 1, 5, 3; column sums 2, 7. Array is:

```
x5
x1
21
```

Problem 27 Given 5 circles. Every 4 have a common point. Prove that there is a point common to all 5.

Problem 28 8 players compete in a tournament. Everyone plays everyone else just once. The winner of a game gets 1, the loser 0, or each gets 1/2 if the game is drawn. The final result is that everyone gets a different score and the player placing second gets the same as the total of the four bottom players. What was the result of the game between the player placing third and the player placing seventh?

Problem 29

(a) The two diagonals of a quadrilateral each divide it into two parts of equal area. Prove it is a parallelogram.

(b) The three main diagonals of a hexagon each divide it into two parts of equal area. Prove they have a common point. [If $ABCDEF$ is a hexagon, then the main diagonals are AD , BE and CF .]

Problem 30 The natural numbers m and n are relatively prime. Prove that the greatest common divisor of $m+n$ and m^2+n^2 is either 1 or 2.

Problem 31 Given a circle c and two fixed points A, B on it. M is another point on c , and K is the midpoint of BM . P is the foot of the perpendicular from K to AM .

- (a) prove that KP passes through a fixed point (as M varies);
 (b) find the locus of P .

Problem 32 Find the smallest value x such that, given any point inside an equilateral triangle of side 1, we can always choose two points on the sides of the triangle, collinear with the given point and a distance x apart.

Problem 33

- (a) A 6×6 board is tiled with 2×1 dominos. Prove that we can always divide the board into two rectangles each of which is tiled separately (with no domino crossing the dividing line).
 (b) Is this true for an 8×8 board?

Problem 34 Given a set of n different positive reals $\{a_1, a_2, \dots, a_n\}$. Take all possible non-empty subsets and form their sums. Prove we get at least $n(n+1)/2$ different sums.

Problem 35 Given a triangle ABC . Let the line through C parallel to the angle bisector of B meet the angle bisector of A at D , and let the line through C parallel to the angle bisector of A meet the angle bisector of B at E . Prove that if DE is parallel to AB , then $CA=CB$.

Problem 36 An infinite arithmetic progression contains a square. Prove it contains infinitely many squares.

Problem 37 Can we label each vertex of a 45-gon with one of the digits 0, 1, ..., 9 so that for each pair of distinct digits i, j one of the 45 sides has vertices labeled i, j ?

Problem 38 Find all real p, q, a, b such that we have $(2x-1)^{20} - (ax+b)^{20} = (x^2+px+q)^{10}$ for all x

Problem 39 We place labeled points on a circle as follows. At step 1, take two points at opposite ends of a diameter and label them both 1. At step $n>1$, place a point at the midpoint of each arc created at step $n-1$ and label it with the sum of the labels at the two adjacent points. What is the total sum of the labels after step n ?
 For example, after step 4 we have: 1, 4, 3, 5, 2, 5, 3, 4, 1, 4, 3, 5, 2, 5, 3, 4.

Problem 40 Given an isosceles triangle, find the locus of the point P inside the triangle such that the distance from P to the base equals the geometric mean of the distances to the sides.

Problem 41 In the triangle ABC , the length of the altitude from A is not less than BC , and the length of the altitude from B is not less than AC . Find the angles.

Problem 42 If m, k, n are natural numbers and $n>1$, prove that we cannot have $m(m+1) = k^n$.

Problem 43 Reduce each of the first billion natural numbers (billion = 10^9) to a single digit by taking its digit sum repeatedly. Do we get more 1s than 2s

Problem 44 Given n odd and a set of integers a_1, a_2, \dots, a_n , derive a new set $(a_1 + a_2)/2, (a_2 + a_3)/2, \dots, (a_{n-1} + a_n)/2, (a_n + a_1)/2$. However many times we repeat this process for a particular starting set we always get integers. Prove that all the numbers in the starting set are equal.

For example, if we started with 5, 9, 1, we would get 7, 5, 3, and then 6, 4, 5, and then 5, 4.5, 5.5. The last set does not consist entirely of integers.

Problem 45

(a) The convex hexagon $ABCDEF$ has all angles equal. Prove that $AB - DE = EF - BC = CD - FA$.

(b) Given six lengths a_1, \dots, a_6 satisfying $a_1 - a_4 = a_5 - a_2 = a_3 - a_6$, show that you can construct a hexagon with sides a_1, \dots, a_6 and equal angles.

Problem 46 Find all possible integer solutions for $\sqrt{x + \sqrt{x \dots (x + \sqrt{x}) \dots}} = y$, where there are 1998 square roots.

Problem 47 ABCD is a convex quadrilateral. A' is the foot of the perpendicular from A to the diagonal BD, B' is the foot of the perpendicular from B to the diagonal AC, and so on. Prove that A'B'C'D' is similar to ABCD.

Problem 48 Find all natural numbers n such that n^2 does not divide $n!$.

Problem 49 Given a lattice of regular hexagons. A bug crawls from vertex A to vertex B along the edges of the hexagons, taking the shortest possible path (or one of them). Prove that it travels a distance at least $AB/2$ in one direction. If it travels exactly $AB/2$ in one direction, how many edges does it traverse?

Problem 50 A circle center O is inscribed in ABCD (touching every side). Prove that angle AOB + angle COD equals 180 degrees.

Problem 51 The natural numbers a, b, n are such that for every natural number k not equal to b , $b - k$ divides $a - k^n$. Prove that $a = b^n$.

Problem 52 How many (algebraically) different expressions can we obtain by placing parentheses in $a_1/a_2/ \dots /a_n$?

Problem 53 What is the smallest number of tetrahedrons into which a cube can be partitioned?

Problem 54

(a) Find the smallest square with last digit not 0 which becomes another square by the deletion of its last two digits.

(b) Find all squares, not containing the digits 0 or 5, such that if the second digit is deleted the resulting number divides the original one.

Problem 55 A circle is inscribed in ABCD. AB is parallel to CD, and $BC = AD$. The diagonals AC, BD meet at E. The circles inscribed in ABE, BCE, CDE, DAE have radius r_1, r_2, r_3, r_4 respectively. Prove that $1/r_1 + 1/r_3 = 1/r_2 + 1/r_4$.

Problem 56 (a) Each of x_1, \dots, x_n is $-1, 0$ or 1 . What is the minimal possible value of the sum of all $x_i x_j$ with $1 \leq i < j \leq n$? (b) Is the answer the same if the x_i are real numbers satisfying $0 \leq |x_i| \leq 1$ for $1 \leq i \leq n$?

Problem 57 Two players have a 3×3 board. 9 cards, each with a different number, are placed face up in front of the players. Each player in turn takes a card and places it on the board until all the cards have been played. The first player wins if the sum of the numbers in the first and third rows is greater than the sum in the first and third columns, loses if it is less, and draws if the sums are equal. Which player wins and what is the winning strategy?

Problem 58 A circle is circumscribed about the triangle ABC. X is the midpoint of the arc BC (on the opposite side of BC to A), Y is the midpoint of the arc AC, and Z is the midpoint of the arc AB. YZ meets AB at D and YX meets BC at E. Prove that DE is parallel to AC and that DE passes through the center of the inscribed circle of ABC.

Problem 59 Bus numbers have 6 digits, and leading zeros are allowed. A number is considered lucky if the sum of the first three digits equals the sum of the last three digits. Prove that the sum of all lucky numbers is divisible by 13.

Problem 60 The beam of a lighthouse on a small rock penetrates to a fixed distance d . As the beam rotates the extremity of the beam moves with velocity v . Prove that a ship with speed at most $v/8$ cannot reach the rock without being illuminated.

Problem 61 A group of 100 people is formed to patrol the local streets. Every evening 3 people are on duty. Prove that you cannot arrange for every pair to meet just once on duty.

Problem 62 A tangent to the inscribed circle of a triangle drawn parallel to one of the sides meets the other two sides at X and Y. What is the maximum length XY, if the triangle has perimeter p ?

Problem 63 The n^2 numbers x_{ij} satisfy the n^3 equations: $x_{ij} + x_{jk} + x_{ki} = 0$. Prove that we can find numbers a_1, \dots, a_n such that $x_{ij} = a_i - a_j$.

Problem 64 Can 1965 points be arranged inside a square with side 15 so that any rectangle of unit area placed inside the square with sides parallel to its sides must contain at least one of the points?

Problem 65 Given n real numbers a_1, a_2, \dots, a_n , prove that you can find n integers b_1, b_2, \dots, b_n , such that the sum of any subset of the original numbers differs from the sum of the corresponding b_i by at most $(n + 1)/4$.

Problem 66 A tourist arrives in Moscow by train and wanders randomly through the streets on foot. After supper he decides to return to the station along sections of street that he has traversed an odd number of times. Prove that this is always possible. [In other words, given a path over a graph from A to B, find a path from B to A consisting of edges that are used an odd number of times in the first path.]

Problem 67 (a) A committee has met 40 times, with 10 members at every meeting. No two people have met more than once at committee meetings. Prove that there are more than 60 people on the committee.

(b) Prove that you cannot make more than 30 subcommittees of 5 members from a committee of 25 members with no two subcommittees having more than one common member.

Problem 68 Given two relatively prime natural numbers r and s , call an integer *good* if it can be represented as $mr + ns$ with m, n non-negative integers and *bad* otherwise. Prove that we can find an integer c , such that just one of $k, c - k$ is good for any k . How many bad numbers are there?

Problem 69 A spy-plane circles point A at a distance 10 km with speed 1000 km/h. A missile is fired towards the plane from A at the same speed and moves so that it is always on the line between A and the plane. How long does it take to hit?

Problem 70 Prove that the sum of the lengths of the edges of a polyhedron is at least 3 times the greatest distance between two points of the polyhedron.

Problem 71 An alien moves on the surface of a planet with speed not exceeding u . A spaceship searches for the alien with speed v . Prove the spaceship can always find the alien if $v > 10u$.

Problem 72 There are an odd number of soldiers on an exercise. The distance between every pair of soldiers is different. Each soldier watches his nearest neighbour. Prove that at least one soldier is not being watched.

Problem 73

(a) B and C are on the segment AD with $AB = CD$. Prove that for any point P in the plane: $PA + PD \geq PB + PC$.

(b) Given four points A, B, C, D on the plane such that for any point P on the plane we have $PA + PD \geq PB + PC$. Prove that B and C are on the segment AD with $AB = CD$.

Problem 74 Can both $x^2 + y$ and $x + y^2$ be squares for x and y natural numbers?

Problem 75 A group of children are arranged into two equal rows. Every child in the back row is taller than the child standing in front of him in the other row. Prove that this remains true if each row is rearranged so that the children increase in height from left to right.

Problem 76 A rectangle $ABCD$ is drawn on squared paper with its vertices at lattice points and its sides lying along the gridlines. $AD = k AB$ with k an integer. Prove that the number of shortest paths from A to C starting out along AD is k times the number starting out along AB .

Problem 77 Given non-negative real numbers a_1, a_2, \dots, a_n , such that $a_{i-1} \leq a_i \leq 2a_{i-1}$ for $i = 2, 3, \dots, n$. Show that you can form a sum $s = b_1 a_1 + \dots + b_n a_n$ with each $b_i = 1$ or -1 , so that $0 \leq s \leq a_1$.

Problem 78 Prove that you can always draw a circle radius A/P inside a convex polygon with area A and perimeter P .

Problem 79 A graph has at least three vertices. Given any three vertices A, B, C of the graph we can find a path from A to B which does not go through C . Prove that we can find two disjoint paths from A to B .

[A graph is a finite set of vertices such that each pair of distinct vertices has either zero or one edges joining the vertices. A path from A to B is a sequence of vertices A_1, A_2, \dots, A_n such that $A=A_1, B=A_n$ and there is an edge between A_i and A_{i+1} for $i = 1, 2, \dots, n-1$. Two paths from A to B are disjoint if the only vertices they have in common are A and B .]

Problem 80 Given a triangle ABC . Suppose the point P in space is such that PH is the smallest of the four altitudes of the tetrahedron $PABC$. What is the locus of H for all possible P ?

Problem 81 Given 100 points on the plane. Prove that you can cover them with a collection of circles whose diameters total less than 100 and the distance between any two of which is more than 1. [The distance between circles radii r and s with centers a distance d apart is the greater of 0 and $d - r - s$.]

Problem 82 The distance from A to B is d kilometers. A plane P is flying with constant speed, height and direction from A to B . Over a period of 1 second the angle PAB changes by α degrees and the angle PBA by β degrees. What is the minimal speed of the plane?

Problem 83 Two players alternately choose the sign for one of the numbers $1, 2, \dots, 20$. Once a sign has been chosen it cannot be changed. The first player tries to minimize the final absolute value of the total and the second player to maximize it. What is the outcome (assuming both players play perfectly)?

Example: the players might play successively: $1, 20, -19, 18, -17, 16, -15, 14, -13, 12, -11, 10, -9, 8, -7, 6, -5, 4, -3, 2$. Then the outcome is 12. However, in this example the second player played badly!

Problem 84 In the acute-angled triangle ABC , AH is the longest altitude (H lies on BC), M is the midpoint of AC , and CD is an angle bisector (with D on AB).

(a) If $AH \leq BM$, prove that the angle $ABC \leq 60$.

(b) If $AH = BM = CD$, prove that ABC is equilateral.

Problem 85

(a) The digits of a natural number are rearranged and the resultant number is added to the original number. Prove that the answer cannot be $99 \dots 9$ (1999 nines).

(b) The digits of a natural number are rearranged and the resultant number is added to the original number to give 10^{10} . Prove that the original number was divisible by 10.

Problem 86 Four lighthouses are arbitrarily placed in the plane. Each has a stationary lamp which illuminates an angle of 90 degrees. Prove that the lamps can be rotated so that at least one lamp is visible from every point of the plane.

Problem 87

(a) Can you arrange the numbers $0, 1, \dots, 9$ on the circumference of a circle, so that the difference between every pair of adjacent numbers is 3, 4 or 5? For example, we can arrange the numbers $0, 1, \dots, 6$ thus: $0, 3, 6, 2, 5, 1, 4$.

(b) What about the numbers $0, 1, \dots, 13$?

Problem 88 Prove that there exists a number divisible by 5^{1000} with no zero digit.

Problem 89 Find all integers x, y satisfying $x^2 + x = y^4 + y^3 + y^2 + y$.

Problem 90 What is the maximum possible length of a sequence of natural numbers x_1, x_2, x_3, \dots such that $x_i \leq 1998$ for $i \geq 1$, and $x_i = |x_{i-1} - x_{i-2}|$ for $i \geq 3$.

Problem 91 499 white rooks and a black king are placed on a 1000×1000 chess board. The rook and king moves are the same as in ordinary chess, except that taking is not allowed and the king is allowed to remain in check. No matter what the initial situation and no matter how white moves, the black king can always:

(a) get into check (after some finite number of moves);

(b) move so that apart from some initial moves, it is always in check after its move;

(c) move so that apart from some initial moves, it is always in check (even just after white has moved).

Prove or disprove each of (a) - (c).

Problem 92 ABCD is a unit square. One vertex of a rhombus lies on side AB, another on side BC, and a third on side AD. Find the area of the set of all possible locations for the fourth vertex of the rhombus.

Problem 93 A natural number k has the property that if k divides n , then the number obtained from n by reversing the order of its digits is also divisible by k . Prove that k is a divisor of 99.

Problem 94 An octagon has equal angles. The lengths of the sides are all integers. Prove that the opposite sides are equal in pairs.

Problem 95 Which is greater: 31^{11} or 17^{14} ? [No calculators allowed!]

Problem 96 A circle radius 100 is drawn on squared paper with unit squares. It does not touch any of the grid lines or pass through any of the lattice points. What is the maximum number of squares it can pass through?

Problem 97 In a group of students, 50 speak English, 50 speak French and 50 speak Spanish. Some students speak more than one language. Prove it is possible to divide the students into 5 groups (not necessarily equal), so that in each group 10 speak English, 10 speak French and 10 speak Spanish.

Problem 98 Prove that:

$$\frac{2}{(x^2 - 1)} + \frac{4}{(x^2 - 4)} + \frac{6}{(x^2 - 9)} + \dots + \frac{20}{(x^2 - 100)} = \frac{11}{(x - 1)(x + 10)} + \frac{11}{(x - 2)(x + 9)} + \dots + \frac{11}{(x - 10)(x + 1)}.$$

Problem 99 The difference between the longest and shortest diagonals of the regular n -gon equals its side. Find all possible n .

Problem 100 The sequence a_n is defined as follows: $a_1 = 1$, $a_{n+1} = a_n + 1/a_n$ for $n \geq 1$. Prove that $a_{100} > 14$.

Problem 101 Given point O inside the acute-angled triangle ABC , and point O' inside the acute-angled triangle $A'B'C'$. D, E, F are the feet of the perpendiculars from O to BC, CA, AB respectively, and D', E', F' are the feet of the perpendiculars from O' to $B'C', C'A', A'B'$ respectively. OD is parallel to $O'A'$, OE is parallel to $O'B'$ and OF is parallel to $O'C'$. Also $OD \cdot O'A' = OE \cdot O'B' = OF \cdot O'C'$. Prove that $O'D'$ is parallel to OA , $O'E'$ to OB and $O'F'$ to OC , and that $O'D' \cdot OA = O'E' \cdot OB = O'F' \cdot OC$.

Problem 102 Prove that any positive integer not exceeding $n!$ can be written as a sum of at most n distinct factors of $n!$.

Problem 103 Given a triangle ABC , and D on the segment AB , E on the segment AC , such that $AD = DE = AC$, $BD = AE$, and DE is parallel to BC . Prove that BD equals the side of a regular 10-gon inscribed in a circle with radius AC .

Problem 104 Given a regular tetrahedron $ABCD$, prove that it is contained in the three spheres on diameters AB, BC and AD . Is this true for any tetrahedron?

Problem 105

(a) Given a 4×4 array with $+$ signs in each place except for one non-corner square which has a $-$ sign. You can change all the signs in any row, column or diagonal. A diagonal can be of any length down to 1. Prove that it is not possible by repeated changes to arrive at all $+$ signs.

(b) What about an 8×8 array?

Problem 106 The medians divide a triangle into 6 smaller triangles. 4 of the circles inscribed in the smaller triangles have equal radii. Prove that the original triangle is equilateral.

Problem 107 Prove that we can find positive integers x, y satisfying $x^2 + x + 1 = py$ for an infinite number of primes p .

Problem 108 9 judges each award 20 competitors a rank from 1 to 20. The competitor's score is the sum of the ranks from the 9 judges, and the winner is the competitor with the lowest score. For each competitor the difference between the highest and lowest ranking (from different judges) is at most 3. What is the highest score the winner could have obtained?

Problem 109 $\{a_i\}$ and $\{b_i\}$ are permutations of $\{1/1, 1/2, \dots, 1/n\}$. $a_1 + b_1 \geq a_2 + b_2 \geq \dots \geq a_n + b_n$. Prove that for every m ($1 \leq m \leq n$) $a_m + a_n \geq 4/m$.

Problem 110 The streets in a city are on a rectangular grid with m east-west streets and n north-south streets. It is known that a car will leave some (unknown) junction and move along the streets at an unknown and possibly variable speed, eventually returning to its starting point without ever moving along the same block twice. Detectors can be positioned anywhere except at a junction to record the time at which the car passes and its direction of travel. What is the minimum number of detectors needed to ensure that the car's route can be reconstructed?

Problem 111 The circle inscribed in the triangle ABC touches the side AC at K . Prove that the line joining the midpoint of AC with the center of the circle bisects the segment BK .

Problem 112 The sequence a_1, a_2, \dots, a_n satisfies the following conditions: $a_1 = 0$, $|a_i| = |a_{i-1} + 1|$ for $i = 2, 3, \dots, n$. Prove that $(a_1 + a_2 + \dots + a_n)/n \geq -1/2$.

Problem 113 The sides and diagonals of $ABCD$ have rational lengths. The diagonals meet at O . Prove that the length AO is also rational.