Local power properties of kernel based goodness of fit tests

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Abstract

If $(X_i, i \in \mathbb{Z})$ is a strictly stationary process with marginal density function f, we are interested in testing the hypothesis $H_0 : \{f = f_0\}$, where f_0 is given. We consider different test statistics based on integrated quadratic forms measuring the proximity between f_n, a kernel estimator of f, and f_0, or between f_n and its expected value computed under H_0. We study the asymptotic local power properties of the testing procedures under local alternatives and discuss the local unbiasedness properties. This study generalizes to the multidimensional case in a context of dependence the corresponding one made by Bickel and Rosenblatt in 1973.

1 Introduction

The goodness of fit tests of $H_0 : \{f = f_0\}$ against the nonparametric alternative hypothesis $H_0^c : \{f \neq f_0\}$, where f_0 is a given density function and f the common marginal density function of the observations, are usually performed under the assumption of independent and identically distributed observations. As noted by Gasser [7], if the stationarity assumption is natural, the independence condition may be too restrictive. Following the idea of Moore [12] and Gleser and Moore [9] who studied the classical goodness of fit tests for dependent data, we extend the approach followed by Bickel and Rosenblatt [1] to the multidimensional dependent case (see also Rosenblatt [16] and Tenreiro [19]).

Bickel and Rosenblatt introduce tests statistics based on the L_2 distance between the kernel density estimator f_n, and either $E_0 f_n$ or f_0, where E_0 is the mathematical expectation under the null hypothesis. These statistics are:

$$I_n^1(\pi) = \int_{\mathbb{R}^d} \{f_n(x) - E_0 f_n(x)\}^2 \pi(x) dx, \tag{1}$$

and

$$I_n^2(\pi) = \int_{\mathbb{R}^d} \{f_n(x) - f_0(x)\}^2 \pi(x) dx, \tag{2}$$

where f_n is a kernel estimator of f based on the observed d-dimensional random variables $X_1, \ldots, X_n$, defined by (cf. Rosenblatt [15] and Parzen [13])

$$f_n(x) = \frac{1}{nh_n^d} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h_n}\right),$$

where K is a kernel, h_n a bandwidth and π a weight function.

Contrary to the case of classical tests based on the empirical distribution function, where there is no multivariate available goodness of fit test (see Révész [14]), we know from the work of Hall [10] that the asymptotic distribution of the statistics $I_n^1(\pi)$ and $I_n^2(\pi)$ can be easily derived in a multidimensional i.i.d. framework by using degenerate U-statistics limit theorems. This approach was used in a recent paper by Fan [6] to generalize the results of Bickel and Rosenblatt [1] derived for $d = 1$.

In order to extend the previous results to the multidimensional dependent case and study local power properties, we consider a strictly stationary process $(X_i, i \in \mathbb{Z})$ whose marginal distribution has a density function f with respect to Lebesgue measure on $\mathbb{R}^d$. In Section 2 we study the asymptotic behaviour of the two statistics $I_n^1(\pi)$ and $I_n^2(\pi)$ both under the null hypothesis H_0 (see Takahata and Yoshihara [17]) and under a sequence of local alternatives $H_{1n} : \{f = f_0 + a_n\gamma + o(a_n)\gamma_n\}$, where (a_n) is a sequence of positive real numbers tending to zero. These results are based on a central limit theorem for degenerate U-statistics corresponding to a geometrically β-mixing process (see Tenreiro [18]).

Under H_0 the two statistics $I_n^j(\pi)$ have asymptotic expansions of the form

$$I_n^j(\pi) \simeq c_j^o(n;\pi) + [d_j^o(n;\pi)]^{-1} Z_j,$$

where the first term is deterministic and the second one is asymptotically normal with zero mean. In Section 3 we develop goodness of fit tests of the hypothesis H_0, by first correcting $I_n^j(\pi)$ for this asymptotic bias, i.e., considering test statistics of the type

$$T_n^j(\pi) = \widehat{d}_j(n;\pi) \left[I_n^j(\pi) - \widehat{c}_j(n;\pi)\right],$$

where $\widehat{c}_j(n;\pi)$ and $\widehat{d}_j(n;\pi)$ are consistent estimators of the deterministic coefficients $c_j^o(n;\pi)$ and $d_j^o(n;\pi)$. For instance based on $I_n^1(\pi)$, we consider the test statistics

$$T_n^{1,1}(\pi) = nh_n^{d/2}\left\{ I_n^1(\pi) - \frac{1}{nh_n^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K^2(u) f_0(x) \pi(x + uh_n) dx du \right\},$$

and

$$T_n^{1,2}(\pi) = nh_n^{d/2}\Big\{ I_n^1(\pi) - \frac{1}{nh_n^d}\frac{1}{n}\sum_{i=1}^n \int_{\mathbb{R}^d} K^2(u)\pi(X_i + uh_n)du \Big\}.$$

Similar test statistics, denoted $T_n^{2,k}(\pi), k = 1, 2$, are derived from $I_n^2(\pi)$. We show that the different test statistics are asymptotically normal under the null hypothesis and we derive associated asymptotically consistent critical regions. We also give asymptotic expansions of these test statistics under local alternative.

Section 4 is the central part of the paper where we study the local power properties of the different tests. For this purpose, we introduce the notions of minimal and maximal limit sequences associated with a consistent test of H_0 against H_0^c asymptotically of level α. They characterize the rates of convergence of the local alternative, such that the asymptotic probability of the critical region under local alternatives is the same as either under H_0 or under a fixed alternative, respectively.

The aim of Section 5 is to discuss the local unbiasedness properties of the testing procedures under local alternatives, and to exhibit some particular local behaviour.

All proofs are gathered in appendices.

2 Asymptotic properties of $I_n^1(\pi)$ and $I_n^2(\pi)$

In this section we study the asymptotic properties of the statistics $I_n^1(\pi)$ and $I_n^2(\pi)$ defined by (1) and (2) respectively. We introduce different assumptions denoted by (K), (B), (π) or (P) on the kernel K, the bandwidth h_n, the weight function π, or the process $(X_i, i \in \mathbb{Z})$.

2.1 First assumptions

Assumptions on the kernel (K)

K is a bounded measurable function on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} K(u)du = 1$.

Assumptions on the bandwidth (B)

We assume that

$$h_n \to 0 \quad \text{and} \quad nh_n^d \to +\infty, \text{ when } n \to +\infty,$$

and there exists $\gamma \in]0,1[$ such that

$$\limsup_n n^\gamma h_n^d < +\infty.$$

The first conditions are usual in kernel estimation theory. The second one is not very restrictive and for instance satisfied if $h_n = O(n^{-\delta})$ for $0 < \delta < 1/d$.

Assumptions on the weight function (π)

The real function π is bounded, nonnegative and almost everywhere continuous on $\mathbb{R}^d$.

2.2 Local alternatives

To define local alternatives we do not consider a unique process, but a sequence of processes $(X_{in}, i \in \mathbb{Z})$, indexed by n (in the sections below we suppress the index n for notational convenience).

Assumptions on the process (P)

For any n, $(X_{in}, i \in \mathbb{Z})$ is a d-dimensional strictly stationary process. This process is β-mixing and we impose some conditions on the uniform convergence to zero of β-mixing coefficients. In particular we suppose that there exist $C > 0$ and $\rho \in]0,1[$ such that

$$\sup_{n \in \mathbb{N}} \beta_n(i) \leq C\rho^i, \ \forall i \in \mathbb{N},$$

where (cf. Volkonskiĭ and Rozanov [20])

$$\beta_n(i) = E\Big[\sup_{A \in F_{in}^{+\infty}} \big| P\big(A | F_{-\infty}^{0n}\big) - P(A) \big| \Big],$$

and $F_{in}^{+\infty}$ (resp. $F_{-\infty}^{0n}$) is the σ-algebra generated by $X_{jn}, j \geq i$ (resp. $X_{jn}, j \leq 0$).

Moreover, we assume that X_{in} and (X_{in}, X_{0n}), $i \geq 1$, have absolutely continuous distributions with pdf g_n and g_{in} respectively, such that

$$\sup_{n,i \in \mathbb{N}} \sup_{\substack{x \in \{u \in \mathbb{R}^d | g_n(u) > 0\} \\ y \in \mathbb{R}^d}} \frac{g_{in}(y,x)}{g_n(x)} < +\infty,$$

and for $x \in \mathbb{R}^d$,

$$g_n(x) = f_0(x) + a_n \gamma(x) + o(a_n)\gamma_n(x), \tag{3}$$

where f_0 is a bounded pdf on $\mathbb{R}^d$, (a_n) is a sequence of positive real numbers tending to zero and

$$\sup_{x\in\mathbb{R}^d} |\gamma(x)| < \infty, \quad \sup_{n\in\mathbb{N}} \sup_{x\in\mathbb{R}^d} |\gamma_n(x)| < \infty,$$
$$\int_{\mathbb{R}^d} |\gamma(x)|dx < \infty, \quad \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} |\gamma_n(x)|dx < \infty.$$

The mixing condition is used to derive a central limit theorem for U-statistics (see Hall [10] and Takahata and Yoshihara [17]). The condition on the conditional density function is satisfied in the i.i.d. case, or for stationary gaussian processes, since the conditional variance is independent of the value of the conditioning variable. Condition (3) gives the interpretation in terms of local alternatives since $\lim_{n\to+\infty} g_n(x) = f_0(x)$, for $x \in \mathbb{R}^d$. The associated rate of convergence is a_n.

At this stage it is interesting to discuss assumption (3) and in particular to give its interpretation in terms of random variables. The following property can be easily derived.

Proposition 2.1 *Let us consider a process $(X_i, i \in \mathbb{Z})$ satisfying the set of assumptions* (P) *with $a_n = 0$, (δ_n) a sequence of positive real numbers tending to zero and $(Z_i, i \in \mathbb{Z})$ a strictly stationary d-dimensional process independent of $(X_i, i \in \mathbb{Z})$ with β-mixing coefficients exponentially decreasing to zero. Then the sequence of processes defined by*

$$X_{in} = X_i + \delta_n Z_i, \; i \in \mathbb{Z},$$

satisfies assumptions (P).

The form of the marginal density function g_n of the process $(X_{in}, i \in \mathbb{Z})$ defined in the previous proposition and its expansion depend on the distribution of Z_0. In particular, it is possible to link the rates of local alternatives in terms of density function (i.e. a_n) and in terms of variables (i.e. δ_n).

If Z_0 is second order integrable, $EZ_0 \neq 0$ and if f_0 has partial continuous derivatives up to order two, which are bounded and integrable on $\mathbb{R}^d$, we have for $x \in \mathbb{R}^d$,

$$\begin{aligned} g_n(x) &= E[f_0(x - \delta_n Z_0)] \\ &= f_0(x) + \delta_n \gamma(x) + \delta_n^2 \gamma_n(x), \end{aligned}$$

where

$$\gamma(x) = -\sum_{i=1}^{d} E[Z_{0i}] \frac{\partial f_0}{\partial x_i}(x)$$

and

$$\gamma_n(x) = \sum_{i,j=1}^{d} E\left[Z_{0i}Z_{0j}\int_0^1 \frac{\partial^2 f_0}{\partial x_i \partial x_j}(x-\delta_n Z_0 t)(1-t)dt\right].$$

If Z_0 is third order integrable, $EZ_0 = 0$ and if f_0 has partial continuous derivatives up to order three, which are bounded and integrable on $\mathbb{R}^d$, we have for $x \in \mathbb{R}^d$,

$$g_n(x) = f_0(x) + \delta_n^2 \gamma(x) + \delta_n^3 \gamma_n(x),$$

where

$$\gamma(x) = \frac{1}{2}\sum_{i,j=1}^{d} E[Z_{0i}Z_{0j}]\frac{\partial^2 f_0}{\partial x_i \partial x_j}(x)$$

and

$$\gamma_n(x) = -\frac{1}{2}\sum_{i,j,k=1}^{d} E\left[Z_{0i}Z_{0j}Z_{0k}\int_0^1 \frac{\partial^3 f_0}{\partial x_i \partial x_j \partial x_k}(x-\delta_n Z_0 t)(1-t)^2 dt\right].$$

In summary if Z_0 is zero mean, we have $a_n = \delta_n$, and otherwise $a_n = \delta_n^2$.

2.3 Asymptotic behaviour of $I_n^1(\pi)$ and $I_n^2(\pi)$

The statistics $I_n^1(\pi)$ and $I_n^2(\pi)$ are related by:

$$\begin{aligned} I_n^2(\pi) &= I_n^1(\pi) + \int_{\mathbb{R}^d} \{E_0 f_n(x) - f_0(x)\}^2 \pi(x)dx \\ &\quad +2\int_{\mathbb{R}^d} \{f_n(x) - E_0 f_n(x)\}\{E_0 f_n(x) - f_0(x)\}\pi(x)dx. \end{aligned}$$

i) Analysis of the bias term $E_0 f_n(x) - f_0(x)$

Let us introduce, for $m \in \mathbb{N}$, the set $W^d(m)$ of pdf with partial continuous derivatives up to order m, which are bounded and integrable on $\mathbb{R}^d$, and the set $K^d(m)$ of kernels of order m, i.e., such that

$$\int_{\mathbb{R}^d} \|x\|^m |K(x)|dx < \infty \text{ and } \int_{\mathbb{R}^d} x_1^{a_1}x_2^{a_2}\dots x_d^{a_d}K(x)dx = 0,$$

for all $a_1, a_2, \dots, a_d \in \mathbb{N}_0$ such that $0 < a_1 + a_2 + \dots + a_d < m$.

If $f_0 \in W^d(m)$ and $K \in K^d(m)$ for some $m \in \mathbb{N}$, we have (see Bosq and Lecoutre [3] pg. 88)

$$E_0 f_n(x) - f_0(x) = h_n^m \Delta_n^m f_0(x), \tag{4}$$

where

$$
\begin{aligned}
&\Delta_n^m f_0(x) \\
= \ & \frac{(-1)^m}{(m-1)!} \sum_{i_1,\ldots,i_m=1}^{d} \int_{\mathbb{R}^d} u_{i_1}\ldots u_{i_m} K(u) \int_0^1 \frac{\partial^m f_0}{\partial x_{i_1}\cdots\partial x_{i_m}}(x-h_n ut)(1-t)^{m-1}dt du.
\end{aligned}
$$

Moreover, by the dominated convergence theorem

$$
\lim_{n\to+\infty} \Delta_n^m f_0(x) = \Delta^m f_0(x),\ x\in\mathbb{R}^d, \tag{5}
$$

with

$$
\Delta^m f_0(x) = \frac{(-1)^m}{m!} \sum_{i_1,\ldots,i_m=1}^{d} \int_{\mathbb{R}^d} u_{i_1}\ldots u_{i_m} K(u)du \, \frac{\partial^m f_0}{\partial x_{i_1}\cdots\partial x_{i_m}}(x). \tag{6}
$$

ii) Asymptotic behaviour of U-statistics

The asymptotic random feature of the statistics $I_n^1(\pi)$ and $I_n^2(\pi)$ essentially depends on the two following U-statistics. The first one is a second order U-statistic defined by

$$
\mathcal{H}_n(\pi) = \frac{2}{n} \sum_{1\le j<i\le n} \{H_n(X_i,X_j) - E_n H_n(X_i,X_j)\},
$$

where for $u, v\in\mathbb{R}^d$,

$$
\begin{aligned}
H_n(u,v) = \frac{1}{h_n^{3d/2}} \int_{\mathbb{R}^d} &\left\{K\left(\frac{x-u}{h_n}\right) - E_n K\left(\frac{x-X_0}{h_n}\right)\right\} \\
&\times\left\{K\left(\frac{x-v}{h_n}\right) - E_n K\left(\frac{x-X_0}{h_n}\right)\right\}\pi(x)dx,
\end{aligned} \tag{7}
$$

and E_n denotes the mathematical expectation under the local alternative. The second U-statistic is

$$
\mathcal{G}_n(\pi) = \frac{2}{\sqrt{n}} \sum_{i=1}^{n} G_n(X_i),
$$

where for $u\in\mathbb{R}^d$,

$$
G_n(u) = \frac{1}{h_n^d} \int_{\mathbb{R}^d} \left\{K\left(\frac{x-u}{h_n}\right) - E_n K\left(\frac{x-X_0}{h_n}\right)\right\}(\Delta_n^m f_0\cdot\pi)(x)dx. \tag{8}
$$

In order to study the asymptotic behaviour of the latter U-statistic, we need the following additional condition where $f_0 \in W^d(m)$ for some $m\in\mathbb{N}$.

Assumption (C)

For any $i \in \mathbb{N}$, there exists $e_i \in \mathbb{R}$ such that for any sequences u_n and v_n on $\mathbb{R}^d$, tending to zero:

$$E_n\left[(\Delta_n^m f_0 \cdot \pi)(X_i + u_n)(\Delta_n^m f_0 \cdot \pi)(X_0 + v_n)\right] \rightarrow e_i, \text{ if } n \rightarrow +\infty.$$

This condition is satisfied if for each $i \in \mathbb{N}$, and under the local alternative, the sequence of pdf $f_{(X_{in}, X_{0n})}$ converges to some pdf $f_{(X_{i\infty}, X_{0\infty})}$ and satisfies the dominated convergence theorem conditions. Then from the almost everywhere continuity of π and the continuity of the derivatives of f_0, we have

$$e_i = E\left[(\Delta^m f_0 \cdot \pi)(X_{i\infty})(\Delta^m f_0 \cdot \pi)(X_{0\infty})\right], \; i \in \mathbb{N}.$$

For independent variables the condition (C) is satisfied with $e_i = 0$ for $i \in \mathbb{N}$.

The following central limit theorem for U-statistics is proven in Appendix A.

Theorem 2.2 *Let us assume that* (K), (B), (π) *and* (P) *are satisfied.*

(i) *The random variable $\mathcal{H}_n(\pi)$ is, under local alternative, asymptotically normal with zero mean and variance given by $2\nu_1^2(\pi)$ where*

$$\nu_1^2(\pi) = \int_{\mathbb{R}^d} f_0^2(x)\pi^2(x)dx \int_{\mathbb{R}^d} (K * \bar{K})^2(z)dz,$$

$\bar{K}(z) = K(-z)$, *and $*$ is the convolution product.*

(ii) *Moreover, if $f_0 \in W^d(m)$ and $K \in K^d(m)$ for some $m \in \mathbb{N}$, and the assumption* (C) *is satisfied, the bivariate random vector $\left(\mathcal{H}_n(\pi), \mathcal{G}_n(\pi)\right)$ is, under local alternative, asymptotically normal with zero mean and diagonal covariance matrix* $\begin{bmatrix} 2\nu_1^2(\pi) & 0 \\ 0 & 4\nu_2^{*2}(\pi) \end{bmatrix}$, *where*

$$\nu_2^{*2}(\pi) = Var_0\left[(\Delta^m f_0 \cdot \pi)(X_0)\right] + 2\sum_{j=1}^{+\infty} \left(e_j - E_0^2\left[(\Delta^m f_0 \cdot \pi)(X_0)\right]\right).$$

In particular under H_0, we have

$$\begin{aligned} \nu_2^{*2}(\pi) &= \nu_2^2(\pi) \text{ (say)} \\ &= Var_0\left[(\Delta^m f_0 \cdot \pi)(X_0)\right] \\ &\quad +2\sum_{j=1}^{+\infty} Cov\left[(\Delta^m f_0 \cdot \pi)(X_j), (\Delta^m f_0 \cdot \pi)(X_0)\right]. \end{aligned}$$

iii) Asymptotic expansions of the statistics

The expansions of the statistics $I_n^1(\pi)$ and $I_n^2(\pi)$ are given in the following result, which is proven in Appendix B.

Theorem 2.3 *Let us assume that* (K), (B), (π) *and* (P) *are satisfied.*

(i) *Under local alternative, we have*

$$\begin{aligned} I_n^1(\pi) &= \frac{1}{nh_n^{d/2}}\mathcal{H}_n(\pi) + a_n^2 \int_{\mathbb{R}^d} \gamma^2(x)\pi(x)dx \\ &+\frac{1}{nh_n^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K^2(u)g_n(x)\pi(x+uh_n)dxdu + o_p(a_n^2) + o_p\left(\frac{1}{nh_n^{d/2}}\right). \end{aligned}$$

(ii) *Under local alternative, if* $f_0 \in W^d(m)$ *and* $K \in K^d(m)$ *for some* $m \in \mathbb{N}$, *we have*

$$\begin{aligned} I_n^2(\pi) &= I_n^1(\pi) + \frac{1}{\sqrt{n}h_n^{-m}}\mathcal{G}_n(\pi) + \int_{\mathbb{R}^d}\{E_0 f_n(x) - f_0(x)\}^2\pi(x)dx \\ &+2h_n^m \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(u)\big(g_n(x) - f_0(x)\big)(\Delta_n^m f_0 \cdot \pi)(x+uh_n)dxdu. \end{aligned}$$

3 Goodness of fit tests

3.1 Some test statistics

From Theorems 2.2 and 2.3 with $a_n = 0$, $I_n^1(\pi)$ and $I_n^2(\pi)$ are asymptotically under the null hypothesis such that

$$I_n^j(\pi) \simeq c_j^o(n;\pi) + [d_j^o(n;\pi)]^{-1} Z_j,$$

where $d_j^o(n;\pi)$ and $c_j^o(n;\pi)$ are deterministic and Z_j are normal variables. Moreover the term $c_j^o(n;\pi)$ is larger than $[d_j^o(n;\pi)]^{-1}$. Therefore to build relevant testing procedures we have first to correct for the asymptotic bias $c_j^o(n;\pi)$, then to introduce the rate $d_j^o(n;\pi)$. The bias corrections consist essentially in substracting $c_j^o(n;\pi)$ or an estimate of this quantity consistent under the null hypothesis. Among the proposed corrections, some just correct for the bias under the null hypothesis, some others under the null and local alternatives.

More precisely we consider the following statistics:

$$T_n^{1,1}(\pi) = nh_n^{d/2}\Big\{ I_n^1(\pi) - \frac{1}{nh_n^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K^2(u) f_0(x)\pi(x+uh_n)dxdu\Big\},$$

$$T_n^{1,2}(\pi) = nh_n^{d/2}\Big\{ I_n^1(\pi) - \frac{1}{nh_n^d}\frac{1}{n}\sum_{i=1}^n \int_{\mathbb{R}^d} K^2(u)\pi(X_i+uh_n)du \Big\},$$

$$T_n^{2,1}(\pi) = d(n)\Big\{ I_n^2(\pi) - \frac{1}{nh_n^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K^2(u)f_0(x)\pi(x+uh_n)dxdu - \int_{\mathbb{R}^d}\{E_0 f_n(x) - f_0(x)\}^2\pi(x)dx \Big\},$$

and

$$T_n^{2,2}(\pi) = d(n)\Big\{ I_n^2(\pi) - \frac{1}{nh_n^d}\frac{1}{n}\sum_{i=1}^n \int_{\mathbb{R}^d} K^2(u)\pi(X_i+uh_n)du - \int_{\mathbb{R}^d}\{E_0 f_n(x) - f_0(x)\}^2\pi(x)dx \Big\},$$

where

$$d(n) = \begin{cases} nh_n^{d/2}, & if\ \lambda \in [0,+\infty[\\ \sqrt{n}\,h_n^{-m}, & if\ \lambda = +\infty \end{cases} \quad with\ \lambda = \lim_{n\to+\infty} nh_n^{d+2m}. \tag{9}$$

3.2 Expansions of the test statistics under local alternatives

Under local alternatives the term

$$\frac{1}{nh_n^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K^2(u)g_n(x)\pi(x+uh_n)dxdu$$

appearing in the expansions of $I_n^1(\pi)$ and $I_n^2(\pi)$ (see Theorem 2.3) may be approximated in different ways. It is first equal to

$$\frac{1}{nh_n^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K^2(u)f_0(x)\pi(x+uh_n)dxdu + \frac{a_n}{nh_n^d}\int_{\mathbb{R}^d}\gamma(x)\pi(x)dx\int_{\mathbb{R}^d}K^2(u)du + o_p\left(\frac{a_n}{nh_n^d}\right),$$

by using the definition of g_n, and it is also equal to

$$\frac{1}{nh_n^d}\frac{1}{n}\sum_{i=1}^n\int_{\mathbb{R}^d}K^2(u)\pi(X_i+uh_n)du + O_p\left(\frac{1}{nh_n^d}\frac{1}{\sqrt{n}}\right),$$

by central limit theorem.

By substituting in the expansions of Theorem 2.3, and by gathering the negligible terms, we get the expansions of the test statistics.

Theorem 3.1 *Let us assume that* (K), (B), (π) *and* (P) *are satisfied.*

(i) *Under local alternative, we have*

$$T_n^{1,2}(\pi) = \mathcal{H}_n(\pi) + nh_n^{d/2}a_n^2\int_{\mathbb{R}^d}\gamma^2(x)\pi(x)dx + o_p\left(nh_n^{d/2}a_n^2\right) + o_p(1),$$

and

$$T_n^{1,1}(\pi) = T_n^{1,2}(\pi) + \frac{a_n}{h_n^{d/2}} \int_{\mathbb{R}^d} K^2(u)du \int_{\mathbb{R}^d} \gamma(x)\pi(x)dx + o\left(\frac{a_n}{h_n^{d/2}}\right) + o_p(1).$$

(ii) *Under local alternative, if $f_0 \in W^d(m)$ and $K \in K^d(m)$ for some $m \in \mathbb{N}$, we have*

$$\begin{aligned} T_n^{2,2}(\pi) \;=\;& \frac{d(n)}{nh_n^{d/2}}\mathcal{H}_n(\pi) + \frac{d(n)}{\sqrt{n}h_n^{-m}}\mathcal{G}_n(\pi) + d(n)a_n^2 \int_{\mathbb{R}^d} \gamma^2(x)\pi(x)dx \\ &+2d(n)h_n^m a_n \int_{\mathbb{R}^d} \Delta^m f_0(x)\gamma(x)\pi(x)dx + o_p\big(d(n)a_n^2\big) + o(d(n)h_n^m a_n) + o_p(1), \end{aligned}$$

and

$$T_n^{2,1}(\pi) = T_n^{2,2}(\pi) + \frac{d(n)a_n}{nh_n^d} \int_{\mathbb{R}^d} \gamma(x)\pi(x)dx \int_{\mathbb{R}^d} K^2(u)du + o\left(\frac{d(n)a_n}{nh_n^d}\right) + o_p(1).$$

3.3 The critical regions

From Theorems 2.2 and 3.1 the different statistics $T_n^{j,k}(\pi)$ are asymptotically normal under the null hypothesis. Moreover for a fixed alternative hypothesis $H_1 : \{f = f_1\}$, with $f_1 \neq f_0$, $T_n^{j,k}(\pi)$ converges in probability to $+\infty$, as soon as the support of the weight function contains the support of f_0 (see Appendix C). Therefore we have to introduce one sided critical regions based on the previous statistics.

The following result extends to the multidimensional and β-mixing process case Theorem 4.1 of Bickel and Rosenblatt [1]. We denote by ϕ the cdf of the standard normal distribution, and by λ the parameter defined in (9).

Theorem 3.2 *Let us assume that* (K), (B), (π) *and* (P) *are satisfied, and that the support of the weight function π contains the support of f_0.*

(i) *The tests associated with the critical regions*

$$C_n^{1,k}(\pi) = \left\{T_n^{1,k}(\pi) \geq \phi^{-1}(1-\alpha)\big(2\nu_1^2(\pi)\big)^{1/2}\right\},$$

with $k = 1, 2$, are asymptotically of level α and consistent to test H_0 against H_0^c.

(ii) *If $f_0 \in W^d(m)$ and $K \in K^d(m)$ for some $m \in \mathbb{N}$, the tests associated with the critical regions*

$$C_n^{2,k}(\pi) = \left\{T_n^{2,k}(\pi) \geq \phi^{-1}(1-\alpha)\big(2\nu_1^2(\pi) + 4\lambda\nu_2^2(\pi)\big)^{1/2}\right\},$$

with $k=1,2$ *and* $\lambda \in [0,+\infty[$, *and*

$$C_n^{2,k}(\pi) = \Big\{ T_n^{2,k}(\pi) \geq 2\phi^{-1}(1-\alpha)|\nu_2(\pi)| \Big\},$$

with $k=1,2$ *and* $\lambda = +\infty$, *are asymptotically of level* α *and consistent to test* H_0 *against* H_0^c.

The asymptotic variance of $\mathcal{G}_n(\pi)/2$ under H_0 has no simple analytic expression, but $\nu_2^2(\pi)$ may be consistently estimated by

$$\begin{aligned}\widehat{\nu}^2_{2,n}(\pi) &= Var_0\big((\Delta^m f_0 \cdot \pi)(X_0)\big) \\ &\quad +2\sum_{j=1}^{m(n)} \Bigg\{ \frac{1}{n-j}\sum_{i=1}^{n-j}(\Delta^m f_0\cdot\pi)(X_i)(\Delta^m f_0\cdot\pi)(X_{i+j}) \\ &\qquad\qquad - \Bigg(\frac{1}{n}\sum_{i=1}^{n}(\Delta^m f_0\cdot\pi)(X_i)\Bigg)^2 \Bigg\},\end{aligned}$$

where $m(n)$ is a sequence of integers tending to $+\infty$. Under H_0, we have

$$\widehat{\nu}^2_{2,n}(\pi) \xrightarrow[n\to+\infty]{p} \nu_2^2(\pi), \text{ if } \quad \frac{m^3(n)}{n} \underset{n\to+\infty}{\longrightarrow} 0.$$

The equivalence properties of the previous tests procedures under local alternative (and under H_0 by imposing $a_n = 0$) are given in the theorem below and are direct consequences of Theorems 3.1 and 3.2.

Theorem 3.3 *Let us assume that the hypothesis* (K), (B), (π) *and* (P) *are satisfied.*

(i) *The tests* $C_n^{1,1}(\pi)$ *and* $C_n^{1,2}(\pi)$ *are asymptotically equivalent under the local alternatives with* $a_n = o\big(h_n^{d/2}\big)$.

Moreover, let us assume that $f_0 \in W^d(m)$ *and* $K \in K^d(m)$ *for some* $m \in \mathbb{N}$.

(ii) *If* $\lambda \in [0,+\infty[$ *(resp.* $\lambda = +\infty$*) the tests* $C_n^{2,1}(\pi)$ *and* $C_n^{2,2}(\pi)$ *are asymptotically equivalent under the local alternatives with* $a_n = o\big(h_n^{d/2}\big)$ *(resp.* $a_n = o\big(\sqrt{nh_n^{2d+2m}}\big)$*).*

(iii) *If* $\lambda = 0$ *the tests* $C_n^{1,1}(\pi)$, $C_n^{1,2}(\pi)$, $C_n^{2,1}(\pi)$ *and* $C_n^{2,2}(\pi)$ *are asymptotically equivalent under the local alternatives with* $a_n = o\big(h_n^{d/2}\big)$.

3.4 Modified test statistics

A number of other test statistics may have been considered by approximating differently the bias term. We discuss some of them, including those proposed by Bickel and Rosenblatt [1], in this subsection.

Let us assume that $f_0 \in W^d(m)$ for some $m \in \mathbb{N}$, and that the kernel K satisfies $\int_{\mathbb{R}^d} \|u\|^m K^2(u)du < \infty$. The dominated convergence theorem and a Taylor expansion provide

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K^2(u) f_0(x)\pi(x+uh_n)dxdu \\
&\quad = \int_{\mathbb{R}^d} K^2(u)du \int_{\mathbb{R}^d} f_0(x)\pi(x)dx + R_0(h_n,m) + o(h_n^m),
\end{aligned}$$

where $R_0(h_n, 0) = 0$, and if $m \geq 1$,

$$R_0(h_n,m) = \sum_{j=1}^{m} \frac{(-1)^j}{j!} h_n^j \sum_{i_1,\ldots,i_j=1}^{d} \int_{\mathbb{R}^d} K^2(u) u_{i_1}\ldots u_{i_j} du \int_{\mathbb{R}^d} \frac{\partial^j f_0}{\partial x_{i_1}\cdots\partial x_{i_j}}(x)\pi(x)dx,$$

with $u = (u_1, u_2, \ldots, u_d)$.

Therefore, the statistics $T_n^{1,3}(\pi)$ and $T_n^{2,3}(\pi)$ defined by

$$T_n^{1,3}(\pi) = nh_n^{d/2}\left\{ I_n^1(\pi) - \frac{1}{nh_n^d}\int_{\mathbb{R}^d} K^2(u)du \int_{\mathbb{R}^d} f_0(x)\pi(x)dx - \frac{1}{nh_n^d} R_0(h_n,m)\right\},$$

and

$$\begin{aligned}
T_n^{2,3}(\pi) = d(n)\Bigg\{ & I_n^2(\pi) - \frac{1}{nh_n^d}\int_{\mathbb{R}^d} K^2(u)du \int_{\mathbb{R}^d} f_0(x)\pi(x)dx \\
& - \frac{1}{nh_n^d} R_0(h_n,m) - \int_{\mathbb{R}^d} \{E_0 f_n(x) - f_0(x)\}^2 \pi(x)dx \Bigg\},
\end{aligned}$$

whenever $m \geq \frac{d}{2}$, and the statistics $T_n^{1,4}(\pi)$ and $T_n^{2,4}(\pi)$ defined by

$$T_n^{1,4} = nh_n^{d/2}\left\{ I_n^1(\pi) - \frac{1}{nh_n^d}\int_{\mathbb{R}^d} K^2(u)du \int_{\mathbb{R}^d} f_0(x)\pi(x)dx - \frac{1}{nh_n^d} R_0(h_n,m-1)\right\},$$

and

$$\begin{aligned}
T_n^{2,4}(\pi) = d(n)\Bigg\{ & I_n^2(\pi) - \frac{1}{nh_n^d}\int_{\mathbb{R}^d} K^2(u)du \int_{\mathbb{R}^d} f_0(x)\pi(x)dx \\
& - \frac{1}{nh_n^d} R_0(h_n,m-1) - \int_{\mathbb{R}^d} \{E_0 f_n(x) - f_0(x)\}^2 \pi(x)dx \Bigg\},
\end{aligned}$$

whenever $m > \frac{d}{2}$, are asymptotically equivalent to $T_n^{1,1}(\pi)$ and $T_n^{2,1}(\pi)$, respectively.

In the unidimensional case $d = 1$ with $m = 1$, $T_n^{1,4}(\pi)$ is considered in Bickel and Rosenblatt [1].

Some other test statistics asymptotically equivalent to the previous ones may be constructed if $nh_n^{d/2+2m} \to 0$ or $nh_n^{2d+2m} \to +\infty$, when $n \to +\infty$. In fact, in these cases we have respectively $\int_{\mathbb{R}^d}\{E_0 f_n(x) - f_0(x)\}^2\pi(x)dx = o\Big(\frac{1}{nh_n^{d/2}}\Big)$ and $\frac{1}{nh_n^d} = o\Big(\frac{1}{\sqrt{n}h_n^{-m}}\Big)$. For instance in the unidimensional case with $m = 2$, the statistic

$$nh_n^{1/2}\Big\{I_n^2(\pi) - \frac{1}{nh_n}\int_{\mathbb{R}} K^2(u)du \int_{\mathbb{R}} f_0(x)\pi(x)dx\Big\},$$

derived from $T_n^{2,3}(\pi)$ or $T_n^{2,4}(\pi)$, is asymptotically equivalent to $T_n^{2,1}(\pi)$ if $h_n = o\big(n^{-\frac{2}{9}}\big)$. This statistic is also considered in [1].

4 Local power analysis

4.1 Minimal and maximal limit sequences

We consider a consistent test of H_0 against H_0^c asymptotically of level α defined by the critical region C_n, and we denote by P_{g_n} the probability under the sequence of local alternative satisfying (P).

Definition 4.1 *The sequence $(\underline{a}_n)$ of positive real numbers, converging to zero when $n \to +\infty$, is a minimal sequence for the test C_n if for any sequence of local alternatives satisfying* (P) *with $\gamma \neq 0$ in* (3)*, we have*

$$\lim_{n\to+\infty} P_{g_n}(\, C_n \,) = \alpha, \ \ if \quad \frac{a_n}{\underline{a}_n} = o(1).$$

Moreover, if the minimal sequence $(\underline{a}_n^I)$ for the test C_n satisfies $\underline{a}_n = O\big(\underline{a}_n^I\big)$ for any minimal sequences $(\underline{a}_n)$ for the test C_n, $(\underline{a}_n^I)$ is called a minimal limit sequence for C_n.

Definition 4.2 *The sequence $(\overline{a}_n)$ of positive real numbers, converging to zero when $n \to +\infty$, is a maximal sequence for the test C_n if for any sequence of local alternatives satisfying* (P) *with $\gamma \neq 0$ in* (3)*, we have*

$$\lim_{n\to+\infty} P_{g_n}(\, C_n \,) = 1, \ \ if \quad \frac{\overline{a}_n}{a_n} = o(1).$$

Moreover, if the maximal sequence $(\overline{a}_n^I)$ for the test C_n satisfies $\overline{a}_n^I = O\big(\overline{a}_n\big)$ for any maximal sequences $(\overline{a}_n)$, for the test C_n, $(\overline{a}_n^I)$ is called maximal limit sequence for C_n.

If there both exist a minimal and a maximal limit sequence for C_n, such that $\underline{a}_n^I = \overline{a}_n^I = a_n^I$ (say), the faster a_n^I is tending to zero, the better is the testing procedure.

4.2 Limit sequences for $C_n^{j,k}(\pi)$, $j,k=1,2$

The limit sequences depend on the testing procedure and for the tests $C_n^{j,k}(\pi), j,k=1,2$, on the choice of the bandwidth h_n. These limit sequences are derived from Theorem 3.1 under the assumption that the weight function is almost everywhere strictly positive on $\mathbb{R}^d$. Then they are particularized to the case of a power function

$$h_n = n^{-a},$$

where a is strictly between 0 and $1/d$ because of assumption (B), which will allow to summarize the results by some figures. In such a case the limit sequences are also power functions

$$\underline{a}_n^I = n^{-\underline{b}} \quad \text{and} \quad \overline{a}_n^I = n^{-\overline{b}},$$

where $\underline{b}$ and $\overline{b}$ are the minimal and maximal limit rates respectively.

We get for the test $C_n^{1,2}(\pi)$ the limit sequence

$$a_n^I = \frac{1}{\sqrt{n}\, h_n^{d/4}},$$

and for the particular case of power functions the limit rates

$$\underline{b} = \overline{b} = \frac{1}{2} - \frac{ad}{4}.$$

For the test $C_n^{1,1}(\pi)$ the minimal and maximal limit sequences do not coincide. We get

$$\underline{a}_n^I = \frac{1}{\sqrt{n}\, h_n^{d/4}} \mathbb{1}_{\left\{n^{-2/3d}=O(h_n)\right\}} + h_n^{d/2} \mathbb{1}_{\left\{h_n=o(n^{-2/3d})\right\}},$$

and

$$\overline{a}_n^I = \frac{1}{\sqrt{n}\, h_n^{d/4}} \mathbb{1}_{\left\{n^{-2/3d}=O(h_n)\right\}} + \frac{1}{nh_n^d} \mathbb{1}_{\left\{h_n=o(n^{-2/3d})\right\}}.$$

For power functions the corresponding limit rates are:

$$\underline{b} = \left(\frac{1}{2} - \frac{ad}{4}\right) \mathbb{1}_{\left\{a \le \frac{2}{3d}\right\}} + \frac{ad}{2} \mathbb{1}_{\left\{a > \frac{2}{3d}\right\}},$$

and

$$\overline{b} = \left(\frac{1}{2} - \frac{ad}{4}\right) \mathbb{1}_{\left\{a \le \frac{2}{3d}\right\}} + \left(1 - ad\right) \mathbb{1}_{\left\{a > \frac{2}{3d}\right\}}.$$

These rates are summarized in Figures 1 and 2. We remark that it is possible to be very close to the parametric rate $\underline{b} = \overline{b} = \frac{1}{2}$ by choosing a large bandwidth ($a \simeq 0$).

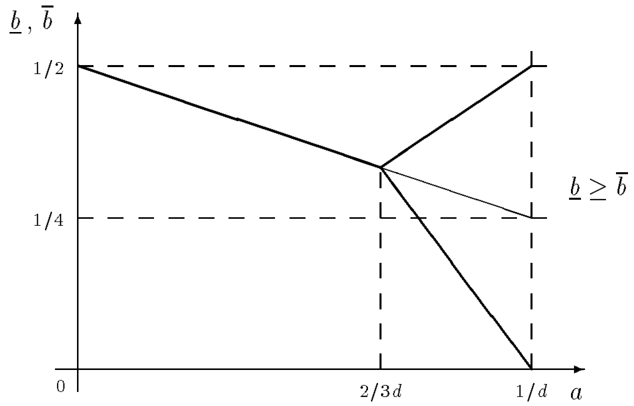

Figure 1: Limit rates for $C_n^{1,1}(\pi)$

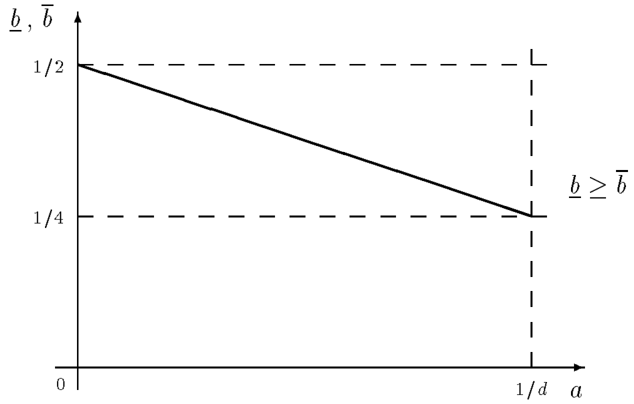

Figure 2: Limit rates for $C_n^{1,2}(\pi)$

Finally we study the asymptotic local power of the tests $C_n^{2,1}(\pi)$ and $C_n^{2,2}(\pi)$ by assuming that $\widehat{\nu}_{2n}^2(\pi)$ is under the sequence of local alternatives a consistent estimator of the variance of $\mathcal{G}_n(\pi)/2$:

$$\widehat{\nu}_{2n}^2(\pi) \underset{n\to+\infty}{\overset{p}{\longrightarrow}} \nu_2^{*2}(\pi).$$

This condition is satisfied if for $i \in \mathbb{N}$ we have

$$E_n\left[(\Delta^m f_0 \cdot \pi)(X_i)(\Delta^m f_0 \cdot \pi)(X_0)\right] \to e_i, n \to +\infty,$$

where $e_i, i \in \mathbb{N}$ is given in assumption (C).

Using the expansion of Theorem 3.1, we get, for the test $C_n^{2,2}(\pi)$, the limit sequences

$$\underline{a}_n^I = \frac{1}{\sqrt{n}} \mathbb{1}_{\left\{n^{-\frac{1}{d+2m}} = O(h_n)\right\}} + \frac{1}{n h_n^{d/2+m}} \mathbb{1}_{\left\{h_n = o(n^{-\frac{1}{d+2m}}) \text{ and } n^{-\frac{1}{d/2+2m}} = O(h_n)\right\}}$$

$$+\frac{1}{\sqrt{n}\,h_n^{d/4}}\mathbb{I}_{\left\{h_n=o(n^{-\frac{1}{d/2+2m}})\right\}},$$

and

$$\overline{a}_n^I = h_n^m\,\mathbb{I}_{\left\{n^{-\frac{1}{d/2+2m}}=O(h_n)\right\}} + \frac{1}{\sqrt{n}\,h_n^{d/4}}\mathbb{I}_{\left\{h_n=o(n^{-\frac{1}{d/2+2m}})\right\}}.$$

The corresponding limit rates are:

$$\underline{b} = \frac{1}{2}\mathbb{I}_{\left\{a\le\frac{1}{d+2m}\right\}} + \big(1-\frac{a}{2}(d+2m)\big)\mathbb{I}_{\left\{\frac{1}{d+2m}<a\le\frac{1}{d/2+2m}\right\}} + \big(\frac{1}{2}-\frac{ad}{4}\big)\mathbb{I}_{\left\{a>\frac{1}{d/2+2m}\right\}},$$

and

$$\overline{b} = am\,\mathbb{I}_{\left\{a\le\frac{1}{d/2+2m}\right\}} + \big(\frac{1}{2}-\frac{ad}{4}\big)\mathbb{I}_{\left\{a>\frac{1}{d/2+2m}\right\}}.$$

Similarly we get for the test $C_n^{2,1}(\pi)$:

i) If $d<2m$:

$$\begin{aligned}\underline{a}_n^I &= \frac{1}{\sqrt{n}}\mathbb{I}_{\left\{n^{-\frac{1}{d+2m}}=O(h_n)\right\}} + \frac{1}{nh_n^{d/2+m}}\mathbb{I}_{\left\{h_n=o(n^{-\frac{1}{d+2m}})\text{ and }n^{-\frac{1}{d/2+2m}}=O(h_n)\right\}}\\ &\quad+\frac{1}{\sqrt{n}\,h_n^{d/4}}\mathbb{I}_{\left\{h_n=o(n^{-\frac{1}{d/2+2m}})\text{ and }n^{-2/3d}=O(h_n)\right\}} + h_n^{d/2}\,\mathbb{I}_{\left\{h_n=o(n^{-2/3d})\right\}},\end{aligned}$$

and

$$\begin{aligned}\overline{a}_n^I &= h_n^m\,\mathbb{I}_{\left\{n^{-\frac{1}{d/2+2m}}=O(h_n)\right\}} + \frac{1}{\sqrt{n}\,h_n^{d/4}}\mathbb{I}_{\left\{h_n=o(n^{-\frac{1}{d/2+2m}})\text{ and }n^{-2/3d}=O(h_n)\right\}}\\ &\quad+\frac{1}{nh_n^{d}}\mathbb{I}_{\left\{h_n=o(n^{-2/3d})\right\}}.\end{aligned}$$

The corresponding limit rates are:

$$\begin{aligned}\underline{b} &= \frac{1}{2}\mathbb{I}_{\left\{a\le\frac{1}{d+2m}\right\}} + \big(1-\frac{a}{2}(d+2m)\big)\mathbb{I}_{\left\{\frac{1}{d+2m}<a\le\frac{1}{d/2+2m}\right\}}\\ &\quad+\big(\frac{1}{2}-\frac{ad}{4}\big)\mathbb{I}_{\left\{\frac{1}{d/2+2m}<a\le\frac{2}{3d}\right\}} + \frac{ad}{2}\mathbb{I}_{\left\{a>\frac{2}{3d}\right\}},\end{aligned}$$

and

$$\overline{b} = am\,\mathbb{I}_{\left\{a\le\frac{1}{d/2+2m}\right\}} + \big(\frac{1}{2}-\frac{ad}{4}\big)\mathbb{I}_{\left\{\frac{1}{d/2+2m}<a\le\frac{2}{3d}\right\}} + \big(1-ad\big)\mathbb{I}_{\left\{a>\frac{2}{3d}\right\}}.$$

ii) If $d\ge 2m$:

$$\begin{aligned}\underline{a}_n^I &= \frac{1}{\sqrt{n}}\mathbb{I}_{\left\{n^{-\frac{1}{d+2m}}=O(h_n)\right\}} + \frac{1}{nh_n^{d/2+m}}\mathbb{I}_{\left\{h_n=o(n^{-\frac{1}{d+2m}})\text{ and }n^{-\frac{1}{d+m}}=O(h_n)\right\}}\\ &\quad+h_n^{d/2}\,\mathbb{I}_{\left\{h_n=o(n^{-\frac{1}{d+m}})\right\}},\end{aligned}$$

and

$$\overline{a}_n^I = h_n^m\,\mathbb{I}_{\left\{n^{-\frac{1}{d+m}}=O(h_n)\right\}} + \frac{1}{nh_n^{d}}\mathbb{I}_{\left\{h_n=o(n^{-\frac{1}{d+m}})\right\}}.$$

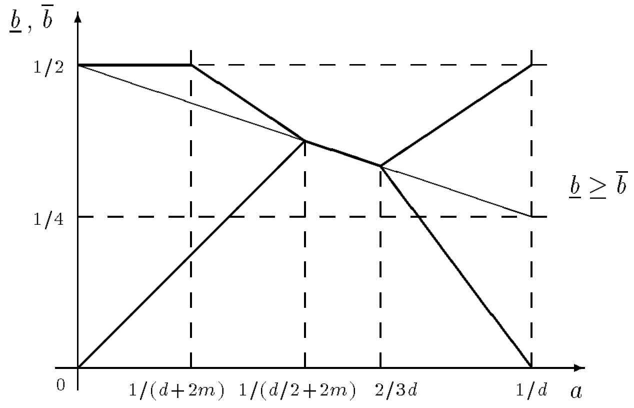

Figure 3: Limit rates for the test $C_n^{2,1}(\pi)$, $d < 2m$

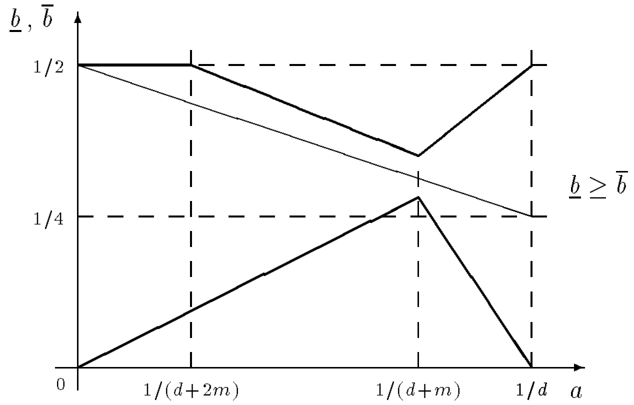

Figure 4: Limit rates for the test $C_n^{2,1}(\pi)$, $d \geq 2m$

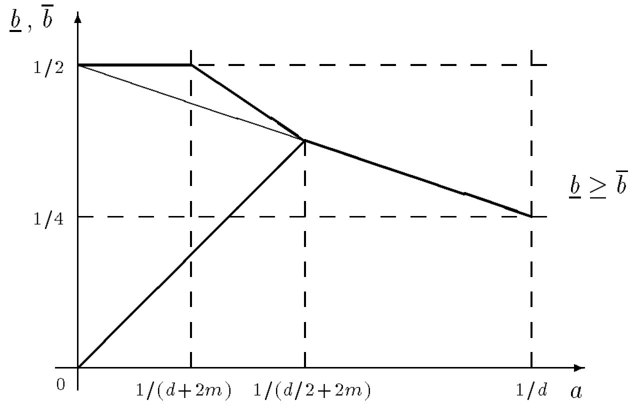

Figure 5: Limit rates for the test $C_n^{2,2}(\pi)$

The corresponding limit rates are:

$$\underline{b} = \frac{1}{2}\mathbb{I}_{\left\{a \leq \frac{1}{d+2m}\right\}} + \left(1 - \frac{a}{2}(d+2m)\right)\mathbb{I}_{\left\{\frac{1}{d+2m} < a \leq \frac{1}{d+m}\right\}} + \frac{ad}{2}\mathbb{I}_{\left\{a > \frac{1}{d+m}\right\}},$$

and

$$\overline{b} = am\mathbb{I}_{\left\{a \leq \frac{1}{d+m}\right\}} + \left(1 - ad\right)\mathbb{I}_{\left\{a > \frac{1}{d+m}\right\}}.$$

These rates are summarized in the Figures 3, 4 and 5.

5 Local unbiasedness properties

The equality of the two limit sequences has interesting implications in terms of local unbiasedness properties of the testing procedures. We discuss this problem, using as a basis the different figures of the previous section. We distinguish three different cases depending on the bandwidth $h_n = n^{-c}$, $0 < c < 1/d$, used to define the goodness of fit tests. They may be described as follows: i) in a neighbourhood of c the two limit rates coincide for c and on the right of c; ii) in a neighbourhood of c the two limit rates coincide for c and on the left of c but they do not coincide on the right of c; and iii) in a neighbourhood of c the two limit rates never coincide.

In the two first cases the asymptotic local power function of the tests $C_n^{j,k}(\pi)$ has the form:

$$\lim_{n\to+\infty} P_{g_n}\left(C_n^{j,k}(\pi)\right) = \begin{cases} \alpha, & \text{if } a_n = o\left(\frac{1}{\sqrt{n}\,h_n^{d/4}}\right) \\ \beta(\gamma), & \text{if } a_n = \frac{1}{\sqrt{n}\,h_n^{d/4}} \\ 1, & \text{if } \frac{1}{\sqrt{n}\,h_n^{d/4}} = o(a_n). \end{cases}$$

In case i), $\beta(\gamma)$ is an increasing function of $\int_{\mathbb{R}^d}\gamma^2(x)\pi(x)dx$ given by

$$\beta(\gamma) = 1 - \phi\left(\phi^{-1}(1-\alpha) - \left(2\nu_1^2(\pi)\right)^{-1/2}\int_{\mathbb{R}^d}\gamma^2(x)\pi(x)dx\right).$$

Then, from the almost everywhere strict positivity imposed to the weight function the tests $C_n^{j,k}(\pi)$ are uniformly locally strictly unbiased (cf. Bickel and Rosenblatt [1]) since we have: $\beta(\gamma) \geq \alpha$ and $\beta(\gamma) > \alpha$ unless $\gamma = 0$, for all bounded and integrable γ functions.

In case ii), $\beta(\gamma)$ is given by

$$\beta(\gamma) = 1 - \phi\Bigg(\phi^{-1}(1-\alpha) - \left(2\nu_1^2(\pi)\right)^{-1/2} \times\left[\int_{\mathbb{R}^d}K^2(u)du\int_{\mathbb{R}^d}\gamma(x)\pi(x)dx + \int_{\mathbb{R}^d}\gamma^2(x)\pi(x)dx\right]\Bigg),$$

and there is a local bias. To get a better understanding of what happens in this case, let us consider $\gamma = \delta\gamma_0$, where γ_0 is a given function and $\delta \in \mathbb{R}$. Let us introduce the function of δ defined by

$$\beta^*(\delta) = \beta(\delta\gamma_0).$$

The function $\beta^*(\cdot)$ is the power function corresponding to the previous alternative. For $\delta^* = \frac{-\int_{\mathbb{R}^d} K^2(u)du \int_{\mathbb{R}^d} \gamma_0(x)\pi(x)dx}{\int_{\mathbb{R}^d} \gamma_0^2(x)\pi(x)dx}$ assumed to be positive, it has the form given in Figure 6, where it can be seen that the problem of bias is only a local one in a neighbourhood of the null hypothesis ($\delta = 0$).

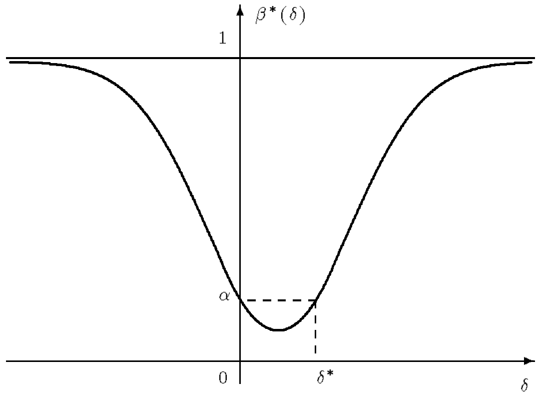

Figure 6: Power function $\beta^*(\cdot)$

Finally, in case iii), i.e., when the minimal and maximal limit sequences do not coincide, the asymptotic local power of the tests $C_n^{j,k}(\pi)$ has the form:

$$\lim_{n\to+\infty} P_{g_n}\left(C_n^{j,k}(\pi)\right) = \begin{cases} \alpha, & \text{if } a_n = o\left(\overline{a}_n^I\right) \\ \beta(\gamma;(a_n)), & \text{if } \overline{a}_n^I = O(a_n) \text{ and } a_n = O(\underline{a}_n^I) \\ 1, & \text{if } \underline{a}_n^I = o(a_n), \end{cases}$$

where $\beta(\gamma;(a_n))$ depends on the position of the alternative. In order to exemplify this situation we will consider the case of the test $C_n^{2,1}(\pi)$ when $0 < c < 1/(d+2m)$. When we consider a local alternative with $a_n = \overline{a}_n^I$, $\beta(\gamma;(a_n))$ may be either 1 or 0 depending on the either positive or negative sign of $\int_{\mathbb{R}^d} \gamma^2(x)\pi(x)dx + 2\int_{\mathbb{R}^d} \Delta^m f_0(x)\gamma(x)\pi(x)dx$. When we consider a local alternative with $\overline{a}_n^I = o(a_n)$ and $a_n = o(\underline{a}_n^I)$, $\beta(\gamma;(a_n))$ may be either 1 or 0 depending on the either positive or negative sign of $\int_{\mathbb{R}^d} \Delta^m f_0(x)\gamma(x)\pi(x)dx$. Finally if we consider a local alternative with $a_n = \underline{a}_n^I$ we have

$$\beta(\gamma;(a_n)) = 1 - \phi\left(\phi^{-1}(1-\alpha) - \left(2\nu_2^2(\pi)\right)^{-1/2} 2\int_{\mathbb{R}^d} \Delta^m f_0(x)\gamma(x)\pi(x)dx\right),$$

which may be either larger or smaller than α depending on the either positive or negative sign of $\int_{\mathbb{R}^d} \Delta^m f_0(x)\gamma(x)\pi(x)dx$.

The lack of local unbiasedness in cases ii) and iii) is a feature of these kernel based testing procedures. It is a consequence of the practice of just correcting the bias of the basic statistics $I_n^1(\pi)$ and $I_n^2(\pi)$ under the null, and not under the local alternative.

APPENDICES

A Asymptotic normality of $(\mathcal{H}_n(\pi), \mathcal{G}_n(\pi))$

Under H_0 and for $m = 2$, Theorem 2.2 was derived by Tenreiro [18] as a consequence of a central limit theorem for degenerate U-statistics generated by a β-mixing process, whose mixing coefficients have an exponential decrease (this result follows the lines of Hall [10] and Takahata and Yoshihara [17], and is based on central limit theorem for triangular array given by Dvoretsky [4]). The introduction of local alternatives does not modify the approach (see Tenreiro [19] for $\mathcal{H}_n(\pi)$). Therefore if the hypotheses (K), (B), (π), (P) and (C) are satisfied, and under local alternative, $(\mathcal{H}_n(\pi), \mathcal{G}_n(\pi))$ is asymptotically normal with zero mean and a diagonal covariance matrix, where

$$\lim_{n\to+\infty} Var_n(\mathcal{H}_n(\pi)) = \lim_{n\to+\infty} E_n[G_{n0}(X_0, \overline{X}_0)],$$

and

$$\lim_{n\to+\infty} Var_n(\mathcal{G}_n(\pi)) = 4\sum_{i=-\infty}^{+\infty} \lim_{n\to+\infty} E_n[G_n(X_i)G_n(X_0)],$$

where $G_{n0}(u,v) = E_n[H_n(X_0,u)H_n(X_0,v)]$, for $u, v \in \mathbb{R}^d$, $\overline{X}_0$ is an independent copy of X_0, and $H_n(\cdot,\cdot)$ and $G_n(\cdot)$ are defined by (7) and (8) respectively.

We compute in the sequel these asymptotic variances.

A.1 Asymptotic variance of $\mathcal{H}_n(\pi)$

From the definition of $H_n(\cdot,\cdot)$ we have for $u, v \in \mathbb{R}^d$,

$$H_n(u,v) \quad = \quad \frac{1}{h_n^{3d/2}} \int_{\mathbb{R}^d} K\left(\frac{x-u}{h_n}\right) K\left(\frac{x-v}{h_n}\right) \pi(x)dx$$

$$
\begin{aligned}
&-\frac{1}{h_n^{3d/2}}\int_{\mathbb{R}^d} K\left(\frac{x-u}{h_n}\right) E_n K\left(\frac{x-X_0}{h_n}\right)\pi(x)dx\\
&-\frac{1}{h_n^{3d/2}}\int_{\mathbb{R}^d} K\left(\frac{x-v}{h_n}\right) E_n K\left(\frac{x-X_0}{h_n}\right)\pi(x)dx\\
&+\frac{1}{h_n^{3d/2}}\int_{\mathbb{R}^d} E_n^2 K\left(\frac{x-X_0}{h_n}\right)\pi(x)dx,
\end{aligned}
$$

where from assumptions (K), (π) and (P) any of the last terms is bounded by

$$
h_n^{d/2}\sup_{x\in\mathbb{R}^d}|\pi(x)|\sup_{n\in\mathbb{N}}\sup_{x\in\mathbb{R}^d} g_n(x)\int_{\mathbb{R}^d}|K|*|\bar{K}|(u)du. \tag{16}
$$

Therefore, uniformly on $u, v\in\mathbb{R}^d$, we have

$$
H_n(u,v)=\frac{1}{h_n^{3d/2}}\int_{\mathbb{R}^d} K\left(\frac{x-u}{h_n}\right)K\left(\frac{x-v}{h_n}\right)\pi(x)dx+O\big(h_n^{d/2}\big), \tag{17}
$$

and then

$$
\begin{aligned}
&\lim_{n\to+\infty} E_n[G_{n0}(X_0,\overline{X}_0)]\\
&= \frac{1}{h_n^{3d}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} K\left(\frac{x-u}{h_n}\right)K\left(\frac{x-v}{h_n}\right)\pi(x)dx\right)^2 g_n(u)g_n(v)dudv.
\end{aligned}
$$

From the form of g_n, and the dominated convergence theorem we get

$$
\lim_{n\to+\infty} Var_n(\mathcal{H}_n(\pi))=\int_{\mathbb{R}^d} f_0^2(x)\pi^2(x)dx\int_{\mathbb{R}^d}(K*\bar{K})^2(z)dz=\nu_1^2(\pi).
$$

A.2 Asymptotic variance of $\mathcal{G}_n(\pi)$

From the definition of $G_n(\cdot)$ we have, for $i=0,1,2,\ldots$,

$$
\begin{aligned}
&E_n\left[G_n(X_i)G_n(X_0)\right]\\
&\quad= E_n\left[\frac{1}{h_n^{2d}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K\left(\frac{x-X_i}{h_n}\right)K\left(\frac{y-X_0}{h_n}\right)(\Delta_n^m f_0\cdot\pi)(x)(\Delta_n^m f_0\cdot\pi)(y)dxdy\right]\\
&\qquad\qquad -E_n^2\left[\frac{1}{h_n^d}\int_{\mathbb{R}^d} K\left(\frac{x-X_0}{h_n}\right)(\Delta_n^m f_0\cdot\pi)(x)dx\right]\\
&\quad= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(u)K(v)E_n\left[(\Delta_n^m f_0\cdot\pi)(X_i+uh_n)(\Delta_n^m f_0\cdot\pi)(X_0+vh_n)\right]dudv\\
&\qquad\qquad -\left(\int_{\mathbb{R}^d} K(u)E_n[(\Delta_n^m f_0\cdot\pi)(X_0+uh_n)]du\right)^2.
\end{aligned}
$$

Then by applying assumption (C), equality (5), the dominated convergence theorem, and by taking into account the form of g_n, we get for $i=1,2,\ldots$

$$
\lim_{n\to+\infty} E_n[G_n(X_i)G_n(X_0)]=e_i-E_0^2[(\Delta^m f_0\cdot\pi)(X_0)],
$$

and for $i = 0$

$$\lim_{n \to +\infty} E_n[G_n(X_0)G_n(X_0)] = Var_0\big[(\Delta^m f_0 \cdot \pi)(X_0)\big].$$

Finally we have

$$\begin{aligned}
\lim_{n \to +\infty} Var_n(\mathcal{G}_n(\pi)) &= 4\Big[Var_0\big[(\Delta^m f_0 \cdot \pi)(X_0)\big] + 2\sum_{i=1}^{+\infty}\big(e_i - E_0^2[(\Delta^m f_0 \cdot \pi)(X_0)]\big)\Big] \\
&= 4\nu_2^{*2}(\pi).
\end{aligned}$$

B Asymptotic expansions of $I_n^1(\pi)$ and $I_n^2(\pi)$

B.1 Decompositions of the statistics

We can first decompose $I_n^1(\pi)$ and $I_n^2(\pi)$ in the following ways

$$\begin{aligned}
I_n^1(\pi) &= \left\{\int_{\mathbb{R}^d}\{f_n(x) - E_n f_n(x)\}^2 \pi(x)dx - E_n \int_{\mathbb{R}^d}\{f_n(x) - E_n f_n(x)\}^2 \pi(x)dx\right\} \\
&\quad + E_n \int_{\mathbb{R}^d}\{f_n(x) - E_n f_n(x)\}^2 \pi(x)dx \\
&\quad + 2\int_{\mathbb{R}^d}\{f_n(x) - E_n f_n(x)\}\{E_n f_n(x) - E_0 f_n(x)\}\pi(x)dx \\
&\quad + \int_{\mathbb{R}^d}\{E_n f_n(x) - E_0 f_n(x)\}^2 \pi(x)dx \\
&= \frac{1}{nh_n^{d/2}} A_n^1 + A_n^2 + 2A_n^3 + A_n^4 \text{ (say)},
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
I_n^2(\pi) &= I_n^1(\pi) + 2\int_{\mathbb{R}^d}\{f_n(x) - E_n f_n(x)\}\{E_0 f_n(x) - f_0(x)\}\pi(x)dx \\
&\quad + 2\int_{\mathbb{R}^d}\{E_n f_n(x) - E_0 f_n(x)\}\{E_0 f_n(x) - f_0(x)\}\pi(x)dx \\
&\quad + \int_{\mathbb{R}^d}\{E_0 f_n(x) - f_0(x)\}^2 \pi(x)dx \\
&= I_n^1(\pi) + \frac{1}{\sqrt{n}h_n^{-m}} B_n^1 + 2B_n^2 + \int_{\mathbb{R}^d}\{E_0 f_n(x) - f_0(x)\}^2 \pi(x)dx \text{ (say)}.
\end{aligned} \tag{19}$$

As usual the idea is to provide expansions for each term of the decompositions.

B.2 Expansion of $I_n^1(\pi)$

Let us consider the decomposition (18).

i) We get

$$\begin{aligned}
A_n^1 &= \frac{1}{n}\sum_{i=1}^{n}\{H_n(X_i,X_i)-E_nH_n(X_i,X_i)\} \\
&\quad +\frac{2}{n}\sum_{1\le j<i\le n}\{H_n(X_i,X_j)-E_nH_n(X_i,X_j)\ \} \\
&= \mathcal{H}_n(\pi)+o_p(1),
\end{aligned}$$

since we easily check that the second order moment of the first term is negligible using (17) and the absolute convergence of the mixing coefficients.

ii) The second term is

$$A_n^2=\frac{1}{nh_n^{d/2}}E_n[H_n(X_0,X_0)]+\frac{2}{n^2h_n^{d/2}}\sum_{1\le j<i\le n}E_n[H_n(X_i,X_j)],$$

where from (16)

$$\begin{aligned}
&E_n[H_n(X_0,X_0)] \\
&= \frac{1}{h_n^{3d/2}}\int_{\mathbb{R}^d}\left\{E_nK^2\left(\frac{x-X_0}{h_n}\right)-E_n^2K\left(\frac{x-X_0}{h_n}\right)\right\}\pi(x)dx \\
&= \frac{1}{h_n^{3d/2}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K^2\left(\frac{x-y}{h_n}\right)g_n(y)dy\pi(x)dx-\frac{1}{h_n^{3d/2}}\int_{\mathbb{R}^d}E_n^2K\left(\frac{x-X_0}{h_n}\right)\pi(x)dx \\
&= \frac{1}{h_n^{d/2}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K^2(u)g_n(y)\pi(y+uh_n)dydu+O\big(h_n^{d/2}\big).
\end{aligned}$$

We now check that the covariance term in negligible. Firstly, from (17) and the assumptions on the pdf g_n and g_{in} we conclude that, for $r>1$, there exists $C>0$ such that

$$\max\left\{\max_{1\le i\le n}E_n^{1/r}|H_n(X_i,X_0)|^r\ ,\ E_n^{1/r}|H_n(\overline{X}_0,X_0)|^r\right\}\le C\big(h_n^d\big)^{\frac{1}{r}-\frac{1}{2}}.$$

From Lemma 1 of Yoshihara [21], we get

$$\begin{aligned}
\left|\frac{1}{n}\sum_{1\le j<i\le n}E_n[H_n(X_i,X_j)]\right| &\le \sum_{i=1}^{n-1}\big|E_n[H_n(X_i,X_0)]\big| \\
&\le 4C\big(h_n^d\big)^{\frac{1}{r}-\frac{1}{2}}\sum_{i=1}^{n-1}\beta_n^{\frac{r-1}{r}}(i),
\end{aligned}$$

where the RHS is negligible for $r<2$ from the assumption on the mixing coefficients.

Therefore

$$A_n^2=\frac{1}{nh_n^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K^2(u)g_n(y)\pi(y+uh_n)dydu+O\left(\frac{1}{nh_n^{d/2}}\right).$$

iii) The third term is

$$\begin{aligned} A_n^3 &= \frac{a_n}{\sqrt{n}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{h_n^d} \int_{\mathbb{R}^d} \left\{ K\left(\frac{x - X_i}{h_n}\right) - E_n K\left(\frac{x - X_0}{h_n}\right) \right\} p_n(x)\pi(x)dx \\ &= \frac{a_n}{\sqrt{n}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} L_n(X_i) \text{ (say)}, \end{aligned}$$

where

$$\begin{aligned} p_n(x) &= a_n^{-1}\{Ef_{nn}(x) - E_0 f_n(x)\} \\ &= \frac{1}{h_n^d} \int_{\mathbb{R}^d} K\left(\frac{x-y}{h_n}\right) \big(\gamma(y) + o(1)\gamma_n(y)\big) dy, \end{aligned}$$

is uniformly bounded using the assumptions on γ and γ_n. Then, since $L_n(\cdot)$ is uniformly bounded and the sequence of mixing coefficients is absolutely convergent, we have

$$A_n^3 = O_p\left(\frac{a_n}{\sqrt{n}}\right).$$

iv) Finally, from the dominated convergence theorem and the assumptions on g_n, we have, for γ continuous,

$$\begin{aligned} A_n^4 &= \int_{\mathbb{R}^d} \{E_n f_n(x) - E_0 f_n(x)\}^2 \pi(x)dx \\ &= \int_{\mathbb{R}^d} \left\{ \frac{1}{h_n^d} \int_{\mathbb{R}^d} K\left(\frac{x-y}{h_n}\right) (g_n(y) - f_0(y))dy \right\}^2 \pi(x)dx \\ &= a_n^2 \int_{\mathbb{R}^d} \left\{ \frac{1}{h_n^d} \int_{\mathbb{R}^d} K\left(\frac{x-y}{h_n}\right) \gamma(y)dy \right\}^2 \pi(x)dx + o\big(a_n^2\big) \\ &= a_n^2 \int_{\mathbb{R}^d} \gamma^2(x)\pi(x)dx + o\big(a_n^2\big). \end{aligned}$$

Density arguments permit us to extend the previous expansion for γ bounded and integrable on $\mathbb{R}^d$.

B.3 Expansion of $I_n^2(\pi)$

Let us consider the decomposition (19) and the equality (4).

i) We have

$$\begin{aligned} 2B_n^1 &= 2\sqrt{n}h_n^{-m} \int_{\mathbb{R}^d} \{f_n(x) - E_n f_n(x)\}\{E_0 f_n(x) - f_0(x)\}\pi(x)dx \\ &= 2\sqrt{n} \int_{\mathbb{R}^d} \{f_n(x) - E_n f_n(x)\}(\Delta_n^m f_0 \cdot \pi)(x)dx \end{aligned}$$

$$
\begin{aligned}
&= 2\sqrt{n}\frac{1}{nh_n^d}\sum_{i=1}^{n}\int_{\mathbb{R}^d}\left\{K\left(\frac{x-X_i}{h_n}\right)-E_nK\left(\frac{x-X_i}{h_n}\right)\right\}(\Delta_n^m f_0\cdot\pi)(x)dx \\
&= \mathcal{G}_n(\pi).
\end{aligned}
$$

ii) The same kind of arguments gives

$$
B_n^2 = h_n^m\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K(u)\big(g_n(x)-f_0(x)\big)(\Delta_n^m f_0\cdot\pi)(x+uh_n)dxdu.
$$

By taking into account all these expansions, we deduce the expansions of $I_n^1(\pi)$ and $I_n^2(\pi)$ given in Theorem 2.3.

C Consistency of the test procedures

In what follows we establish the consistency of the tests based on critical regions $C_n^{j,k}(\pi)$ by proving that $T_n^{j,k}(\pi) \xrightarrow[n\to+\infty]{p} +\infty$, $j,k=1,2$, for a fixed alternative hypothesis H_1 : $\{f=f_1\}$, with $f_1\neq f_0$.

If f_1 is the density of the observed process $(X_i, i\in\mathbb{Z})$, we have

$$
\begin{aligned}
I_n^1(\pi) &= 2\int_{\mathbb{R}^d}\{f_n(x)-E_1f_n(x)\}\{E_1f_n(x)-E_0f_n(x)\}\pi(x)dx \\
&\quad+\int_{\mathbb{R}^d}\{f_n(x)-E_1f_n(x)\}^2\pi(x)dx+\int_{\mathbb{R}^d}\{E_1f_n(x)-E_0f_n(x)\}^2\pi(x)dx,
\end{aligned}
$$

where E_1 is the mathematical expectation under the hypothesis H_1. From points ii) and iv) of Appendix B, the first two terms converge to zero in probability, when $n\to+\infty$, and

$$
\int_{\mathbb{R}^d}\{E_1f_n(x)-E_0f_n(x)\}^2\pi(x)dx=\int_{\mathbb{R}^d}\{f_1(x)-f_0(x)\}^2\pi(x)dx+o(1).
$$

Therefore, for $k=1,2$ we have

$$
\begin{aligned}
\frac{T_n^{1,k}(\pi)}{nh_n^{d/2}} &= I_n^1(\pi)-\frac{1}{nh_n^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K^2(u)f_0(x-uh_n)\pi(x)dxdu \\
&= \int_{\mathbb{R}^d}\{f_1(x)-f_0(x)\}^2\pi(x)dx+o_p(1).
\end{aligned}
$$

If $f_0\in W^d(m)$ and $K\in K^d(m)$, the same kind of arguments and the equality

$$
\int_{\mathbb{R}^d}\{E_0f_n(x)-f_0(x)\}^2\pi(x)dx=h_n^{2m}\int_{\mathbb{R}^d}\left(\Delta_n^m f_0(x)\right)^2dx \text{ (cf. (4))},
$$

gives for $k = 1, 2$,

$$\frac{T_n^{2,k}(\pi)}{d(n)} = \int_{\mathbb{R}^d} \{f_1(x) - f_0(x)\}^2 \pi(x) dx + o_p(1).$$

The conclusion follows from the convergence $nh_n^{d/2} \underset{n \to +\infty}{\longrightarrow} +\infty$ and $d(n) \underset{n \to +\infty}{\longrightarrow} +\infty$, and the fact that $\int_{\mathbb{R}^d} \{f_1(x) - f_0(x)\}^2 \pi(x) dx > 0$, since the support of the weight function π contains the support of f_0.

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