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A Laplace Transform Piecewise Linearized Method for a Second Order Hyperbolic Equation

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Abstract. Numerical solutions of a one-dimensional second order hyperbolic equation are presented. The numerical method consists of applying the Laplace transform to remove the time dependent terms in the governing equation and boundary conditions. A second-order linear ordinary differential equation is obtained which is discretized using a piecewise linearized method. We present some numerical tests to show the convergence and efficiency of the numerical method.

Keywords: Hyperbolic equation, Laplace transform, piecewise linearized
PACS: 02.60.Jh, 02.60.Lj (AMS classification 35L20, 65M12, 65M22)

INTRODUCTION

One of the main difficulties encountered in the numerical solutions of hyperbolic problems is the numerical oscillations in the vicinity of sharp discontinuities [3]. The objective of this work is to apply a Laplace transform-piecewise-linearized method that is in good agreement with the analytical solution and does not exhibit numerical oscillations. For that purpose, let us consider the hyperbolic telegraph equation

$$\frac{\partial c}{\partial t} + \theta \frac{\partial^2 c}{\partial t^2} = \frac{\partial}{\partial x}(Pc) + D\frac{\partial^2 c}{\partial x^2}, \ x \in \,]a,b[, t > 0, \tag{1}$$

where c is the mass concentration, D is the diffusion coefficient, P is the velocity field and $\theta \in \,]0,1]$ is the parameter that measures the propagation speed of the mass wave and can be regarded as the relaxation time of the mass flux. Note that for $\theta = 0$, equation (1) is the classical parabolic advection-diffusion equation. For our problem we consider the initial conditions given by

$$c(x,0) = c_0(x), \quad \theta \frac{\partial c}{\partial t}(x,0) = c_1(x), \qquad x \in [a,b] \tag{2}$$

and the boundary conditions

$$c(a,t) = f(t), \qquad c(b,t) = g(t), \qquad t > 0. \tag{3}$$

Problems similar to (1)–(3), which describes a non-Fickian diffusion process, can be found in previous works such as [2, 3, 4, 5, 7].

THE NUMERICAL METHOD

In this section we describe the numerical method applied to solve the problem (1)–(3). Our approach can be separated in three steps. First, we apply the Laplace transform to (1)–(3) in order to remove the time dependent terms and we obtain an ordinary differential equation in x that also depends on the Laplace transform parameter s. Secondly, we solve the ordinary differential equation obtained using a space discretization: the piecewise linearized method similar to the method suggested in [6]. Lastly, using a numerical inverse Laplace transform algorithm, described in [1, 2], we obtain the final approximate solution. There are two main reasons to use the Laplace transform technique. First, it gives accurate solutions at a specific time, and we do not need to do computations in the time domain using time steps. Secondly, we avoid undesirable numerical oscillations, that are related with the bad choices of time steps.

We denote the Laplace transform of the mass concentration c by $\widetilde{c}$. We obtain the ordinary differential equation

$$\frac{d^2\widetilde{c}}{dx^2} - \lambda_s^2 \widetilde{c} + \frac{d}{dx}\left(\frac{P}{D}\widetilde{c}\right) = -\frac{c_0(x)}{D}(1+\theta s) - \frac{c_1(x)}{D} \tag{4}$$

where $\lambda_s = (s(\theta s+1)/D)^{1/2}$ and s is a complex variable, with the boundary conditions, derived from (3), $\widetilde{c}(a,s) = \widetilde{f}(s)$ and $\widetilde{c}(b,s) = \widetilde{g}(s)$. For non-constant P we must solve (4) with a spatial discretization. The approximate solution of c is obtained by using an inverse Laplace transform algorithm.

We consider a piecewise linearized method for discretizing the ordinary differential equation (4). Assume we have a space discretization $x_i = a + i\Delta x$, $i = 0,\ldots,N$, where $\Delta x = (b-a)/N$. Let $\widetilde{C}_i(s)$, $i = 0,\ldots,N$ represent the approximation of $\widetilde{c}(x_i,s)$ in the Laplace transform domain. After the spatial discretization we obtain the linear system

$$K(s)\widetilde{\mathbf{C}}(s) = \widetilde{b}(s), \tag{5}$$

where $K(s) = [K_{i,j}(s)]$ is a band matrix of size $(N-1)\times(N-1)$, $\widetilde{\mathbf{C}}(s) = [\widetilde{C}_1(s),\ldots,\widetilde{C}_{N-1}(s)]^T$ and $\widetilde{b}(s)$ contains source terms and boundary conditions. In our case we obtain a matrix K with bandwidth three.

In what follows, we present the space discretization by giving the entries of the matrix K and the vector $\widetilde{b}$. As we have seen above the matrix K and the numerical approximation of the grid point depend on s. For the sake of clarity we omit the parameter s denoting $K_{i,j}(s)$ and $\widetilde{C}_i(s)$ by $K_{i,j}$ and $\widetilde{C}_i$ respectively.

Following the suggestions presented in [6], for a slightly different problem, we can rewrite equation (4) in the form

$$\frac{d^2\widetilde{c}}{dx^2} + \frac{P}{D}\frac{d\widetilde{c}}{dx} + \left(\frac{P'}{D} - \lambda_s^2\right)\widetilde{c} = -\frac{c_0(x)}{D}(1+\theta s) - \frac{c_1(x)}{D}, \tag{6}$$

where P' represents the x derivative of P. In each interval $[x_{i-1},x_{i+1}]$ the equation can be approximated by

$$\frac{d^2\widetilde{C}}{dx^2} + \frac{P_i}{D}\frac{d\widetilde{C}}{dx} + \left(\frac{P'_i}{D} - \lambda_s^2\right)\widetilde{C} = -\frac{c_0(x_i)}{D}(1+\theta s) - \frac{c_1(x_i)}{D}, \tag{7}$$

where $\widetilde{C}(x,s)$ represents an approximation of $\widetilde{c}(x,s)$, $P_i = P(x_i)$ and $P'_i = P'(x_i)$. This equation is obtained from (6) by freezing the coefficients at the mid-point of the considered interval. The solution of (7) in $[x_{i-1},x_{i+1}]$ is

$$\widetilde{C}(x,s) = A_i e^{v_{s,i}^{+}(x-x_i)} + B_i e^{v_{s,i}^{-}(x-x_i)} + \widetilde{C}_p(x_i,s), \tag{8}$$

with $v_{s,i}^{\pm} = -\frac{P_i}{2D} \pm \sqrt{\left(\frac{P_i}{2D}\right)^2 + \left(\lambda_s^2 - \frac{P'_i}{D}\right)}$ and $\widetilde{C}_p(x_i,s) = \frac{-c_0(x_i)(1+\theta s)-c_1(x_i)}{P'_i - s(1+\theta s)}$ is a particular solution.

For simplicity of notation, and as we have done already for $\widetilde{C}(x,s)$, we denote $\widetilde{C}_p(x_i,s)$ by $\widetilde{C}_{p_i}$ in what follows. The values A_i and B_i can be determined from (8) as

$$B_i = \widetilde{C}_i - A_i - \widetilde{C}_{p_i}, \tag{9}$$

$$\widetilde{C}_{i-1} = A_i e^{-v_{s,i}^{+}\Delta x} + (\widetilde{C}_i - A_i - \widetilde{C}_{p_i})e^{-v_{s,i}^{-}\Delta x} + \widetilde{C}_{p_{i-1}}, \tag{10}$$

$$\widetilde{C}_{i+1} = A_i e^{v_{s,i}^{+}\Delta x} + (\widetilde{C}_i - A_i - \widetilde{C}_{p_i})e^{v_{s,i}^{-}\Delta x} + \widetilde{C}_{p_{i+1}}. \tag{11}$$

From (10) and (11), by equaling the values of A_i, we obtain the following three-point finite difference equations, for $i = 1,\ldots,N-1$,

$$K_{i,i-1}\widetilde{C}_{i-1} + K_{i,i}\widetilde{C}_i + K_{i,i+1}\widetilde{C}_{i+1} = \widetilde{C}_{p_i}(K_{i,i-1} + K_{i,i} + K_{i,i+1}), \tag{12}$$

where

$$K_{i,i-1} = e^{(v_{s,i}^{+}+v_{s,i}^{-})\Delta x}, \qquad K_{i,i} = -e^{v_{s,i}^{+}\Delta x} - e^{v_{s,i}^{-}\Delta x} \qquad \text{and} \qquad K_{i,i+1} = 1.$$

The vector $\widetilde{b}(s)$ is represented by

$$\begin{bmatrix} \widetilde{C}_{p_1}(K_{1,0}+K_{1,1}+K_{1,2}) \\ \widetilde{C}_{p_2}(K_{2,1}+K_{2,2}+K_{2,3}) \\ \vdots \\ \widetilde{C}_{p_{N-2}}(K_{N-2,N-3}+K_{N-2,N-2}+K_{N-2,N-1}) \\ \widetilde{C}_{p_{N-1}}(K_{N-1,N-2}+K_{N-1,N-1}+K_{N-1,N}) \end{bmatrix} - \begin{bmatrix} K_{1,0}\widetilde{C}_0 \\ 0 \\ \vdots \\ 0 \\ K_{N-1,N}\widetilde{C}_N \end{bmatrix}.$$

The next step is to determine an approximate solution $C(x_i,t)$ from $\widetilde{C}(x_i,s)$ by using the Laplace inversion numerical method described in [1, 2].

The errors that come from the numerical inversion of Laplace transform are described in [2]. We can prove the spatial discretization error, using the piecewise linearized method, is at least of second order and similarly to what was done in [2] we obtain a second order error convergence for the full numerical method.

NUMERICAL TESTS

In order to see the performance of the Laplace transform-piecewise-linearized method (Laplace-PL), presented in the previous section, we compare it with the Laplace transform-finite-differences method (Laplace-FD) described in [2]. We start to choose a problem with $\theta = 0$, for which we are able to compute the exact solution, to analyze the order of convergence of the numerical methods. Secondly, we choose a problem, with $\theta \neq 0$, with a discontinuous initial condition to see how the numerical methods handle and propagate the discontinuity. For this non-Fickian problem we present one example for constant P and another one with P non-constant.

Let us consider the problem with $\theta = 0$ and P constant, that is,

$$\frac{\partial c}{\partial t} = P\frac{\partial c}{\partial x} + \frac{\partial^2 c}{\partial x^2}, \quad x \in]0,\infty[, t > 0. \tag{13}$$

For the initial condition $c(x,0) = 0$ and the boundary conditions $c(0,t) = 1$, $c(\infty,t) = 0$, the analytical solution is given by

$$c(x,t) = \frac{1}{2}\left(\text{erfc}\left[\frac{x+Pt}{2\sqrt{t}}\right] + e^{-Px}\text{erfc}\left[\frac{x-Pt}{2\sqrt{t}}\right]\right). \tag{14}$$

To have information about the space discretization errors we show the global error in Figure 1 for the two schemes. We observe the Laplace-PL method has a smaller error than the Laplace-FD method. In Figure 1, for $P = 2$, the rate of convergence for the Laplace-FD method is 2.0 and for the Laplace-PL method is 2.6. Additionally, for $P = -2$, the Laplace-FD method has a rate of convergence of order 1.9 and the Laplace-PL has a rate of convergence 2.4. Therefore, we can observe that for our experiments the Laplace-PL method seems to be more accurate.
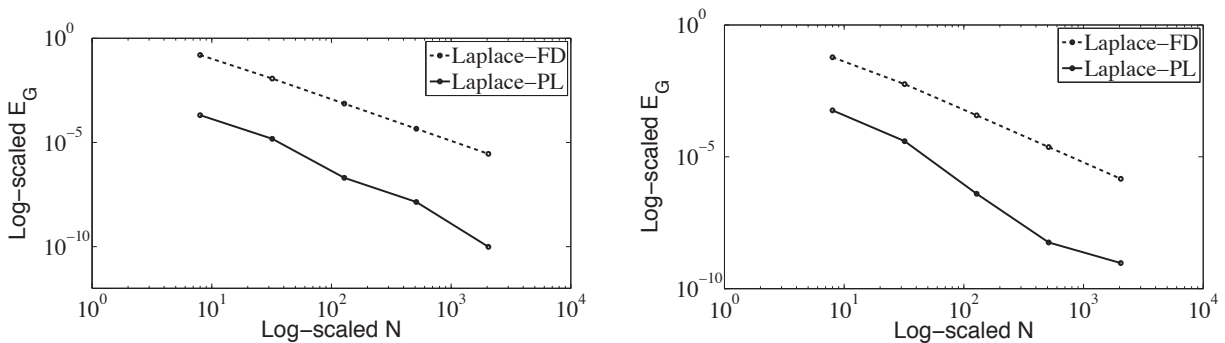
FIGURE 1. Rate of convergence of the numerical methods for $\theta = 0$, $t = 1$, where the number of discrete points N is plotted versus the global error $E_G = \max_i |c(x_i,t) - C(x_i,t)|$. Left figure: $P = 2$ and $0 \leq x \leq 10$. Right figure: $P = -2$ and $0 \leq x \leq 12$.

Now we turn to the hyperbolic problem, for $\theta \neq 0$, given by

$$\frac{\partial c}{\partial t} + \theta\frac{\partial^2 c}{\partial t^2} = \frac{\partial}{\partial x}(Pc) + \frac{\partial^2 c}{\partial x^2}, \quad x \in]0,\infty[, t > 0, \tag{15}$$

with initial conditions $c(x,0) = 0$, $\theta\frac{\partial c}{\partial t}(x,0) = 0$ and boundary conditions $c(0,t) = 1$, $c(\infty,t) = 0$.

In Figure 2 we display the results for constant P. We observe the Laplace-PL method performs better than the Laplace-FD method, since for this method it is evident that oscillations are not avoided for small space steps near the discontinuity.

For non-constant P we show the behaviour of the solution in Figure 3, for $P(x) = -2e^{-x}$ and $P(x) = 2x$. In both cases oscillations are present for the Laplace-FD method.
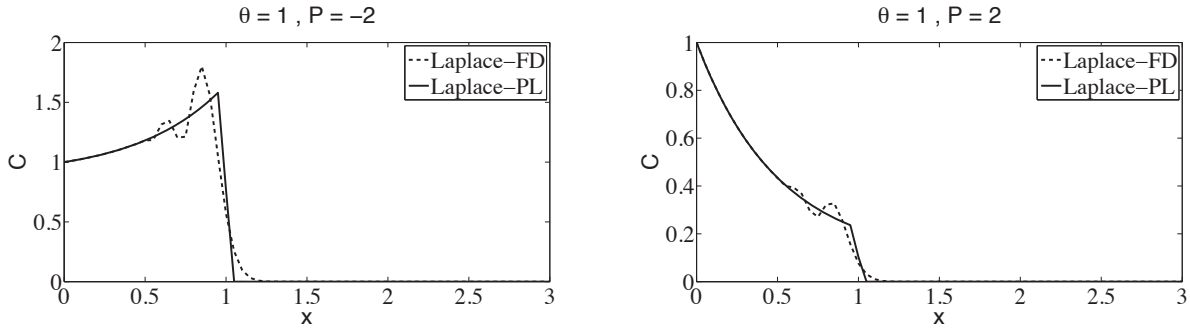
2189

FIGURE 2. Approximate solution using different space discretizations, for different values of P and $\theta = 1$ at $t = 1$ for $\Delta x = 0.05$.
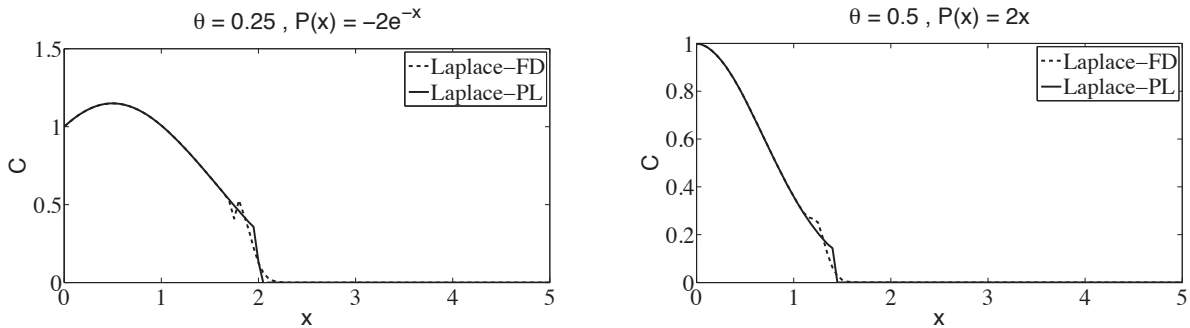

FIGURE 3. Approximate solution using different space discretizations, for different values of P and θ at $t = 1$ for $\Delta x = 0.05$.

FINAL REMARKS

We have studied a differential equation which depends on the parameters θ and P. The equation is parabolic for $\theta = 0$ and therefore the solution is smooth; for $\theta \neq 0$ the equation is hyperbolic and therefore transports initial discontinuities. If P is non-constant, the spatial discretization is mandatory. We applied the Laplace transform in time to obtain an ordinary differential equation and for the spatial discretization of this equation we used a piecewise linearized method. Then, we applied the inverse Laplace algorithm to the approximate solution of the ordinary differential equation. The accuracy of the Laplace transform-piecewise-linearized method has been assessed since it provides smooth solutions, avoiding oscillations near discontinuities where other methods do not.

REFERENCES

1. J. Ahn, S. Kang, Y. Kwon, A flexible inverse Laplace transform algorithm and its application, *Computing* **71(2)**, 115–131 (2003).
2. A. Araújo, A.K. Das, C. Neves, E. Sousa, Numerical solution for a non-Fickian diffusion in a periodic potential, to appear in *Communications in Computational Physics*, 2012.
3. H.-T. Chen, K.-C. Liu, Analysis of non-Fickian diffusion problems in a composite medium, *Computer Physics Communications* **150**, 31–42 (2003).
4. J. Crank, *The Mathematics of Diffusion*, Oxford University Press, 1974.
5. A.K. Das, A non-Fickian diffusion equation, *J. Appl. Phys* **70**, 1355–1358 (1991).
6. J.I. Ramos, On the numerical treatment of an ordinary differential equation arising in one-dimensional non-Fickian diffusion problems, *Computer Physics Communications* **170**, 231–238 (2005).
7. O.D.L. Strack, A mathematical model for dispersion with a moving front in groundwater, *Water Resour. Res.* **28(11)**, 2973–2980 (1992).