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A second order explicit finite difference method for the fractional advection diffusion equation

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ABSTRACT

We develop a numerical method for fractional advection diffusion problems with source terms in domains with homogeneous boundary conditions. The numerical method is derived by using a Lax–Wendroff-type time discretization procedure, it is explicit and second order accurate. The convergence of the numerical method is studied and numerical results are presented.

1. Fractional advection diffusion equation

In many applications, an equation commonly used to describe transport diffusive problems is the classical advection diffusion (or dispersion) equation. The classical advection diffusion equation uses second-order Fickian diffusion which is based on the assumption that solute particles undergo an addition of successive increments that are independent, where identically distributed random variables have finite variance and the distribution of the sum of such increments is a normal distribution. Therefore, the fundamental solutions of the classical advection diffusion equation are Gaussian densities.

The anomalous diffusion [1,2] extends the capabilities of models built on the stochastic process of Brownian motion, which can be described by Lévy motion which assumes that significant deviations from the mean can occur, where large jumps are more frequent than in the Brownian motion.

The fractional advection diffusion equation was firstly proposed by Chaves [3] to investigate the mechanism of super-diffusion and with the goal of having a model able to generate the Lévy distribution. It was given by

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} = D\left(\frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial^\alpha u}{\partial(-x)^\alpha}\right), \tag{1}$$

where u is the concentration, V is the average velocity, x is the spatial coordinate, t is the time, D is the diffusion coefficient, α is the order of the fractional differentiation with $1 < \alpha \le 2$. The fractional advection diffusion equation was later generalized by Benson et al. [4,5], to include a parameter β, given by

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} = D\left(\frac{1}{2} + \frac{\beta}{2}\right)\frac{\partial^\alpha u}{\partial x^\alpha} + D\left(\frac{1}{2} - \frac{\beta}{2}\right)\frac{\partial^\alpha u}{\partial(-x)^\alpha}, \tag{2}$$

where β is the relative weight of solute particle forward versus backward transition probability. For $-1 \le \beta \le 0$, the transition probability is skewed backward, while for $0 \le \beta \le 1$ the transition probability is skewed forward. For $\beta = 0$, we obtain the model presented in [3], that is, the transition of the solute particles is symmetric.

We consider an equation with a source term, which can be expressed as follows

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} = D\nabla^\alpha_\beta u + p(x,t), \tag{3}$$

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where the fractional operator is given by

$$\nabla_\beta^\alpha u = \frac{1}{2}(1+\beta)\frac{\partial^\alpha u}{\partial x^\alpha} + \frac{1}{2}(1-\beta)\frac{\partial^\alpha u}{\partial(-x)^\alpha}. \quad (4)$$

The interest in numerical methods for problems involving advection and fractional diffusion has been increasing in the last years. Different tools have been applied, such as finite difference methods [6–8], finite volume methods [9], spectral methods [10] and finite element methods for linear and nonlinear diffusion problems [11–17], just to name a few. However, concerning the use of finite differences the numerical methods presented in the literature, which are second order in time and space are in general implicit methods. In this paper, we develop an explicit numerical method which is second order in time and space for fractional advection diffusion problems with source terms in unbounded and bounded domains with homogeneous boundary conditions, by using a Lax–Wendroff-type time discretization procedure. Since the numerical method is explicit, it is a more cost effective method than the implicit schemes. Additionally explicit methods are better tools for problems where advection plays an important role. The classical Lax–Wendroff method was derived for hyperbolic equations [18] and afterwards was extended for advection diffusion equations. This method uses a small stencil in time and also uses extensively the original differential equation, that is, the discretization procedure converts time derivatives in space derivatives.

Let us consider the problem defined in a domain $\mathbb{R}$ with an initial condition

$$u(x,0) = f(x), \quad x \in \mathbb{R}$$

and boundary conditions

$$\lim_{x\to-\infty} u(x,t) = 0 \quad \text{and} \quad \lim_{x\to\infty} u(x,t) = 0.$$

If we have a bounded domain we assume

$$u(a,t) = 0 \qquad u(b,t) = 0,$$

since the problem can equally be considered in the whole real line by taking an extension where we have $u(x,t) = 0$, for $x \le a$ and $x \ge b$.

The usual way of representing the fractional derivatives is by the Riemann–Liouville formula. The Riemann–Liouville fractional derivatives of order α, for $x \in [a,b]$, $-\infty \le a < b \le \infty$, are defined by

$$\frac{\partial^\alpha u}{\partial x^\alpha}(x,t) = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_a^x u(\xi,t)(x-\xi)^{n-\alpha-1}d\xi, \quad n-1<\alpha<n \quad (5)$$

$$\frac{\partial^\alpha u}{\partial(-x)^\alpha}(x,t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_x^b u(\xi,t)(\xi-x)^{n-\alpha-1}d\xi, \quad n-1<\alpha<n \quad (6)$$

where $\Gamma(\cdot)$ is the Gamma function and $n = [\alpha]+1$, with $[\alpha]$ denoting the integer part of α. Properties about the fractional derivatives can be found for instance in [19–24].

The remainder of this paper is organized as follows. In Section 2, we describe the numerical method, including its matricial form and in Section 3 we study the convergence based in the consistency and the stability analysis. The fourth section includes some numerical tests which confirm the second order convergence of the numerical method and we end with some final remarks.

2. Construction and implementation of the scheme

2.1. Discretization of the fractional operator

In this section, a numerical approximation to the Riemann–Liouville derivative is presented. We describe shortly the approximation, which is second order accurate [25,26].

Consider first the left derivative, that is,

$$\frac{\partial^\alpha u}{\partial x^\alpha}(x,t) = \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_{-\infty}^x u(\xi,t)(x-\xi)^{1-\alpha}d\xi, \quad 1<\alpha<2. \quad (7)$$

We consider the discretization domain $x_j = j\Delta x,\ j \in \mathbb{Z}$. Let

$$\mathcal{I}_\alpha^l(x) = \int_{-\infty}^x u(\xi,t)(x-\xi)^{1-\alpha}d\xi. \quad (8)$$

First, the following approximation at x_j is done,

$$\frac{d^2}{dx^2}\mathcal{I}_\alpha^l(x_j) \simeq \frac{1}{\Delta x^2}[\mathcal{I}_\alpha^l(x_{j-1}) - 2\mathcal{I}_\alpha^l(x_j) + \mathcal{I}_\alpha^l(x_{j+1})]. \quad (9)$$

Secondly, we compute these integrals by approximating the function u by a linear spline $s_j^l(\xi)$, whose nodes and knots are chosen at x_k, $k \le j$, that is, an approximation to (8) becomes

$$I_\alpha^l(x_j) = \int_{-\infty}^{x_j} {s_j}^l(\xi)(x_j - \xi)^{1-\alpha} d\xi, \tag{10}$$

where the spline $s_j^l(\xi)$ interpolates $\{u(x_j, t),\ k \le j\}$ in the interval $(-\infty, x]$. We obtain,

$$I_\alpha^l(x_j) = \frac{1}{(2-\alpha)(3-\alpha)} \Delta x^{2-\alpha} \sum_{k=-\infty}^{j} u(x_k, t) a_{j,k}^l, \tag{11}$$

where the $a_{j,k}^l$ are defined by

$$a_{j,k}^l = \begin{cases} (j-k+1)^{3-\alpha} - 2(j-k)^{3-\alpha} + (j-k-1)^{3-\alpha}, & k \le j-1 \\ 1, & k = j. \end{cases} \tag{12}$$

Therefore, from (9)–(12), we have an approximation of (7) given by

$$\begin{aligned} \frac{d^2}{dx^2} \mathcal{I}_\alpha^l(x_j) &\simeq \frac{1}{\Delta x^2} \frac{1}{\Gamma(2-\alpha)} [I_\alpha^l(x_{j-1}) - 2I_\alpha^l(x_j) + I_\alpha^l(x_{j+1})] \\ &\simeq \frac{\Delta x^{2-\alpha}}{\Delta x^2 \Gamma(4-\alpha)} \left[\sum_{k=-\infty}^{j-1} u(x_k, t) a_{j-1,k}^l - 2 \sum_{k=-\infty}^{j} u(x_k, t) a_{j,k}^l + \sum_{k=-\infty}^{j+1} u(x_k, t) a_{j+1,k}^l \right]. \end{aligned}$$

Finally, an approximation to (7) can be given by $\frac{\delta_l^\alpha u(x_j,t)}{\Delta x^\alpha}$, where

$$\delta_l^\alpha u(x_j, t) = \frac{1}{\Gamma(4-\alpha)} \sum_{k=-\infty}^{j+1} q_{j,k}^l u(x_k, t), \tag{13}$$

for

$$\begin{aligned} q_{j,k}^l &= a_{j-1,k}^l - 2a_{j,k}^l + a_{j+1,k}^l, \quad k \le j-1 \\ q_{j,j}^l &= -2a_{j,j}^l + a_{j+1,j}^l \\ q_{j,j+1}^l &= a_{j+1,j+1}^l. \end{aligned} \tag{14}$$

We can write it as,

$$\delta_l^\alpha u(x_j, t) = \frac{1}{\Gamma(4-\alpha)} \sum_{m=-1}^{\infty} q_{j,j-m}^l u(x_{j-m}, t). \tag{15}$$

Let us know consider the right derivative

$$\frac{\partial^\alpha u}{\partial(-x)^\alpha}(x, t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_x^\infty u(\xi, t)(\xi - x)^{1-\alpha} d\xi, \quad 1 < \alpha < 2. \tag{16}$$

Similarly for the right derivative, let

$$\mathcal{I}_\alpha^r(x) = \int_x^\infty u(\xi, t)(\xi - x)^{1-\alpha} d\xi. \tag{17}$$

First, we do the following approximation at x_j,

$$\frac{d^2}{dx^2} \mathcal{I}_\alpha^r(x_j) \simeq \frac{1}{\Delta x^2} [\mathcal{I}_\alpha^r(x_{j-1}) - 2\mathcal{I}_\alpha^r(x_j) + \mathcal{I}_\alpha^r(x_{j+1})]. \tag{18}$$

Similarly we approximate (17) by considering a linear spline, $s_j^d(\xi)$, which interpolates $\{u(x_k, t),\ k \ge j\}$ in the interval $[x, \infty)$. We obtain

$$I_\alpha^r(x_j) = \int_{x_j}^\infty s_j^r(\xi)(\xi - x_j)^{1-\alpha} d\xi = \frac{\Delta x^{2-\alpha}}{(2-\alpha)(3-\alpha)} \sum_{k=j}^{\infty} u(x_k, t) a_{j,k}^r, \tag{19}$$

where

$$a^r_{j,k} = \begin{cases} (k-j+1)^{3-\alpha} - 2(k-j)^{3-\alpha} + (k-j-1)^{3-\alpha}, & j+1 \le k \\ 1, & k = j. \end{cases} \tag{20}$$

Therefore, from (18)–(20), we have an approximation given by

$$\begin{aligned} \frac{d^2}{dx^2}\mathcal{I}^r_\alpha(x_j) &\simeq \frac{1}{\Delta x^2}\frac{1}{\Gamma(2-\alpha)}[I^r_\alpha(x_{j-1}) - 2I^r_\alpha(x_j) + I^r_\alpha(x_{j+1})] \\ &\simeq \frac{\Delta x^{2-\alpha}}{\Delta x^2\Gamma(4-\alpha)}\left[\sum_{k=j-1}^{\infty} u(x_k,t)a^r_{j-1,k} - 2\sum_{k=j}^{\infty} u(x_k,t)a^r_{j,k} + \sum_{k=j+1}^{\infty} u(x_k,t)a^r_{j+1,k}\right]. \end{aligned}$$

Finally, an approximation of (16) can be given by $\frac{\delta^\alpha_r u(x_j,t)}{\Delta x^\alpha}$, where

$$\delta^\alpha_r u(x_j,t) = \frac{1}{\Gamma(4-\alpha)}\sum_{k=j-1}^{\infty} q^r_{j,k} u(x_k,t), \tag{21}$$

for

$$\begin{aligned} q^r_{j,k} &= a^r_{j-1,k} - 2a^r_{j,k} + a^r_{j+1,k}, \quad k \ge j+1 \\ q^r_{j,j} &= -2a^r_{j,j} + a^r_{j-1,j} \\ q^r_{j,j-1} &= a^r_{j-1,j-1}. \end{aligned} \tag{22}$$

We can write it as,

$$\delta^\alpha_r u(x_j,t) = \frac{1}{\Gamma(4-\alpha)}\sum_{m=-\infty}^{-1} q^r_{j,j-m} u(x_{j-m},t). \tag{23}$$

Let us re-write the previous fractional operators. For that, let us define,

$$a_m = \begin{cases} (m+1)^{3-\alpha} - 2m^{3-\alpha} + (m-1)^{3-\alpha}, & m \ge 1 \\ 1, & m = 0 \end{cases} \tag{24}$$

and

$$q_m = \begin{cases} a_{m-1} - 2a_m + a_{m+1}, & m \ge 1 \\ -2a_0 + a_1, & m = 0 \\ a_0, & m = -1. \end{cases} \tag{25}$$

We can rewrite the fractional operators in the following way,

$$\delta^\alpha_l u(x_j,t) = \frac{1}{\Gamma(4-\alpha)}\sum_{m=-1}^{\infty} q_m u(x_{j-m},t), \tag{26}$$

$$\delta^\alpha_r u(x_j,t) = \frac{1}{\Gamma(4-\alpha)}\sum_{m=-1}^{\infty} q_m u(x_{j+m},t). \tag{27}$$

Note that we can extend easily all of these results to a bounded domain for which $u(a,t) = u(b,t) = 0$. In this case, we consider an extension to the whole real line by assuming $u(x,t) = 0$ for $x \le a$ and $x \ge b$.

2.2. Numerical method

To derive a finite difference scheme we suppose there are approximations $\mathbf{U}^n := \{U^n_j\}$ to the values $u(x_j,t_n)$ at the mesh points

$$x_j = j\Delta x, \quad j \in \mathbb{Z} \quad \text{and} \quad t_n = n\Delta t, \quad n \ge 0,$$

where Δx denotes the uniform space step and Δt the uniform time step. Let

$$\nu = \frac{V\Delta t}{\Delta x} \quad \text{and} \quad \mu_\alpha = \frac{D\Delta t}{\Delta x^\alpha}.$$

The quantity ν is called the Courant (or CFL) number and μ_α is associated with the diffusion coefficient.

We derive a finite difference scheme using Taylor expansions. Let us expand u about time level n, that is, $t = n\Delta t$, to obtain

$$u(x, t_{n+1}) - u(x, t_n) = \Delta t \frac{\partial u}{\partial t}(x, t_n) + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2}(x, t_n) + \mathcal{O}(\Delta t^3). \tag{28}$$

Then, from (3),

$$\frac{\partial^2 u}{\partial t^2}(x, t) = -V \frac{\partial^2 u}{\partial x \partial t}(x, t) + D\nabla_\beta^\alpha \left(\frac{\partial u}{\partial t}(x, t) \right) + p_t(x, t) \tag{29}$$

$$\frac{\partial^2 u}{\partial x \partial t}(x, t) = -V \frac{\partial^2 u}{\partial x^2}(x, t) + D \frac{\partial}{\partial x}[\nabla_\beta^\alpha u(x, t)] + p_x(x, t). \tag{30}$$

Therefore, a spatial finite-difference approximation can be obtained by dropping out the $\alpha + 1$ and higher-spatial-derivative terms, from the previous equalities, holding

$$\frac{\partial^2 u}{\partial t^2}(x, t) \simeq V^2 \frac{\partial^2 u}{\partial x^2}(x, t) - Vp_x(x, t) + p_t(x, t). \tag{31}$$

Inserting (3) and (31) into (28) gives

$$\begin{aligned} u(x, t_{n+1}) \simeq\ & u(x, t_n) + \Delta t \left(-V \frac{\partial u}{\partial x}(x, t_n) + D\nabla_\beta^\alpha u(x, t_n) + p(x, t_n) \right) \\ & + \frac{1}{2} \Delta t^2 \left(V^2 \frac{\partial^2 u}{\partial x^2}(x, t_n) - Vp_x(x, t) + p_t(x, t) \right). \end{aligned} \tag{32}$$

Therefore (32) can be written in the form

$$u(x, t_{n+1}) \simeq u(x, t_n) - V \Delta t \frac{\partial u}{\partial x}(x, t_n) + \Delta t D \nabla_\beta^\alpha u(x, t_n) + \frac{1}{2} \Delta t^2 V^2 \frac{\partial^2 u}{\partial x^2}(x, t_n) + \Delta t \tilde{p}(x, t), \tag{33}$$

where

$$\tilde{p}(x, t) = p(x, t) + \frac{1}{2} \Delta t(-Vp_x(x, t) + p_t(x, t)).$$

The difference $(u^{n+1} - u^n)$ becomes

$$u(x, t_{n+1}) - u(x, t_n) = U_j^{n+1} - U_j^n. \tag{34}$$

Let us define the following operators,

$$\Delta_0 U_j^n = \frac{1}{2}(U_{j+1}^n - U_{j-1}^n), \qquad \delta^2 U_j^n = U_{j+1}^n - 2U_j^n + U_{j-1}^n \tag{35}$$

and the fractional operator

$$\delta_\beta^\alpha u(x_j, t_n) = \frac{1}{2}(1 + \beta)\delta_l^\alpha u(x_j, t_n) + \frac{1}{2}(1 - \beta)\delta_r^\alpha u(x_j, t_n). \tag{36}$$

If we discretize the first derivative with the central difference operator, the second order derivative with second order difference operator and the fractional derivative with the respective fractional difference operator, from (33), we have

$$U_j^{n+1} = U_j^n - \nu \Delta_0 U_j^n + \mu_\alpha \delta_\beta^\alpha U_j^n + \frac{1}{2} \nu^2 \delta^2 U_j^n + \Delta t \tilde{p}_j^n, \tag{37}$$

where $\tilde{p}_j^n = \tilde{p}(x_j, t_n)$.

If we do not know the source term and only know a discrete set of values, we need to approximate the values of the partial derivatives. Therefore we can use the forward difference in time to approximate $p_t(x, t)$ and the central difference in space to approximate $p_x(x, t)$. In this case we obtain

$$\begin{aligned} \tilde{p}(x_j, t_n) &\approx p_j^n + \frac{1}{2} \Delta t \left(-V \frac{p_{j+1}^n - p_{j-1}^n}{2\Delta x} + \frac{p_j^{n+1} - p_j^n}{\Delta t} \right) \\ &= \frac{1}{2}(p_j^{n+1} + p_j^n) - \frac{\nu}{4}(p_{j+1}^n - p_{j-1}^n), \end{aligned}$$

where $p_j^n = p(x_j, t_n)$.

2.3. Matricial form

For a better understanding on how to implement the numerical method we give the matricial form of the method. We have the numerical scheme

$$U_j^{n+1} = U_j^n - \nu \Delta_0 U_j^n + \frac{1}{2}\nu^2 \delta^2 U_j^n + \mu_\alpha \delta_\beta^\alpha U_j^n + \Delta t \tilde{p}_j^n,$$

where

$$\delta_\beta^\alpha U_j^n = \frac{1}{2}(1+\beta)\delta_l^\alpha U_j^n + \frac{1}{2}(1-\beta)\delta_r^\alpha U_j^n. \tag{38}$$

To build the matricial form we assume our solution has compact support. Therefore, the numerical method can be written in the matricial form

$$\mathbf{U}^{n+1} = M\mathbf{U}^n + \mathbf{b}^n + \mathbf{p}^n \tag{39}$$

where $\mathbf{U}^n = [U_1^n \cdots U_{N-1}^n]^T$, $\mathbf{p}^n = [\Delta t \tilde{p}_1^n \cdots \Delta t \tilde{p}_{N-1}^n]^T$, $\mathbf{b}^n$ contains the boundary values, and M is an $(N-1) \times (N-1)$ matrix given by

$$M = A + \frac{1}{2\Gamma(4-\alpha)}\mu_\alpha B$$

where A and B are $(N-1) \times (N-1)$ matrices. The matrix A is related with the advection and the matrix B with diffusion. The matrix B is given by

$$B = (1+\beta)Q + (1-\beta)Q^T$$

where

$$Q = \begin{bmatrix} q_0 & q_{-1} & 0 & \cdots & & 0 \\ q_1 & & & & & \\ & & & \ddots & & \vdots \\ \vdots & & \ddots & & & 0 \\ & & & & & q_{-1} \\ q_{N-2} & & \cdots & & q_1 & q_0 \end{bmatrix}.$$

The vector $\mathbf{b}^n$ is composed of two parts $\mathbf{b}^n = \mathbf{b}_A^n + \mathbf{b}_B^n$, where the vector $\mathbf{b}_A^n$ contains the boundary values related to the matrix A and the vector $\mathbf{b}_B^n$ contains the boundary values related to the matrix B. We have $\mathbf{b}^n = \mathbf{0}$, since we are assuming $U_0^n = U_N^n = 0$. The matrix A is given by

$$A = \begin{bmatrix} 1-\nu^2 & \nu(\nu-1)/2 & & & \\ \nu(\nu+1)/2 & & \ddots & & \\ & \ddots & & & \nu(\nu-1)/2 \\ & & & \nu(\nu+1)/2 & 1-\nu^2 \end{bmatrix}.$$

3. Convergence analysis

In this section we analyze the convergence of the numerical method using the framework of consistency and stability. We have the global error given by $E^n = u^n - U^n$, where u^n and U^n are respectively exact and approximate solutions. Therefore,

$$E^{n+1} = ME^n + \Delta t T^n,$$

where the matrix M contains the coefficients of the difference formulas and T^n is the truncation error. Then

$$E^{n+1} = M^{n+1}E_0 + \Delta t \sum_{k=0}^{n} T^k M^{n-k}.$$

We have

$$\|E^{n+1}\| \le \|M^{n+1}\| \, \|E_0\| + \Delta t \sum_{k=0}^{n} \|T^k\| \, \|M^{n-k}\|.$$

If $\|M^k\| \le C$, for all $0 \le k \le n+1$, it follows,

$$\|E^{n+1}\| \le C\|E_0\| + (n+1)\Delta t C \max_{0\le k\le n} \|T^k\|.$$

Regarding the consistency, let $u = u(x, t)$ be the exact solution and let us assume $p(x, t) = 0$. The truncation error at each discrete point x_j, is given by

$$\begin{aligned} T_j^n &= \frac{u_j^{n+1} - u_j^n}{\Delta t} + V\frac{u_{j+1}^n - u_{j-1}^n}{\Delta t} - V^2\frac{\Delta t}{2}\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} - \frac{D}{2\Delta x^\alpha}\delta_\beta^\alpha u_j^n \\ &= \left(\frac{\partial u}{\partial t}\right)_j^n + \frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_j^n + \mathcal{O}(\Delta t^2) + V\left(\frac{\partial u}{\partial x}\right)_j^n + \mathcal{O}(\Delta x^2) \\ &\quad - V^2\frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n + \mathcal{O}(\Delta x^2) - D(\nabla_\beta^\alpha u)_j^n + \mathcal{O}(\Delta x^2), \end{aligned}$$

since the fractional operator is $\mathcal{O}(\Delta x^2)$ [25,26]. From (31), it follows that

$$T_j^n = \left(\frac{\partial u}{\partial t}\right)_j^n + \mathcal{O}(\Delta t^2) + V\left(\frac{\partial u}{\partial x}\right)_j^n + \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta x^2) - D(\nabla_\beta^\alpha u)_j^n + \mathcal{O}(\Delta x^2),$$

and the numerical method has an order of accuracy close to $\mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2)$.

In order to derive stability conditions for the finite difference schemes, we apply the von Neumann analysis or Fourier analysis. Note that if the scheme is von Neumann stable then the finite difference method is Lax stable in the 2-norm, which means $\|M^n\|_2$ is bounded for all n [27]. Fourier analysis assumes that we have a solution defined in the whole real line. If u_j^n is the exact solution $u(x_j, t_n)$, let U_j^n be a perturbation of u_j^n. The perturbation error

$$e_j^n = U_j^n - u_j^n \tag{40}$$

will be propagated forward in time according to the equation

$$e_j^{n+1} = e_j^n - \nu\Delta_0 e_j^n + \frac{1}{2}\nu^2\delta^2 e_j^n + \mu_\alpha\delta_\beta^\alpha e_j^n. \tag{41}$$

The von Neumann analysis assumes that any finite mesh function, such as, the error e_j^n will be decomposed into a Fourier series with terms given by $\kappa_p^n e^{i\xi_p(j\Delta x)}$, where κ_p^n is the amplitude of the p-th harmonic. The product $\xi_p\Delta x$ is often called the phase angle $\theta = \xi_p\Delta x$ and covers the domain $[-\pi, \pi]$.

Considering a single mode $\kappa^n e^{ij\theta}$, its time evolution is determined by the same numerical scheme as the error e_j^n. Hence inserting a representation of this form into a numerical scheme we obtain stability conditions. The stability conditions will be satisfied if the amplitude factor κ does not grow in time, that is, if we have $|\kappa(\theta)| \le 1, \quad$ for all θ.

The next lemma characterizes the coefficients q_m, defined by (25), and it follows straightforward from some of the properties presented in [26].

Lemma 1. *Consider the coefficients q_m defined by (25). Then, for $1 < \alpha \le 2$,*

(a) *We have $q_{-1} = 1$, $q_0 = 2^{3-\alpha} - 4 \le 0$, $q_1 = 3^{3-\alpha} - 4 \times 2^{3-\alpha} + 6$, which can be positive or negative depending on the value of α, that is, $q_1 \le 0$ for $\alpha \le 1.5545$ and $q_1 \ge 0$ otherwise. Also*

$$q_m = (m+2)^{3-\alpha} - 4(m+1)^{3-\alpha} + 6m^{3-\alpha} - 4(m-1)^{3-\alpha} + (m-2)^{3-\alpha} \ge 0, \quad m \ge 2.$$

(b) $\lim_{m\to\infty} q_m = 0$, $q_{-1} + q_1 \ge 0$ *and* $q_{m+1} \le q_m \le q_2$.
(c) $\sum_{m=2}^{\infty} q_m = -q_{-1} - q_0 - q_1 = -3 + 3 \times 2^{3-\alpha} - 3^{3-\alpha}$ *and* $\sum_{m=-1}^{\infty} q_m = 0$.

First we consider the case $\beta = 0$.

Theorem 2. *For $\beta = 0$, the numerical method is von Neumann stable if, and only if,*

$$\nu^2 - \mu_\alpha\frac{1}{2\Gamma(4-\alpha)}\sum_{m=-1}^{\infty}(-1)^m q_m \le 1, \quad 1 < \alpha \le 2. \tag{42}$$

Proof. For $\beta = 0$ we have,

$$\kappa(\theta) = 1 - i\nu\sin\theta + \nu^2(\cos\theta - 1) + \frac{\mu_\alpha}{\Gamma(4-\alpha)}\sum_{m=-1}^{\infty} q_m\cos(m\theta),$$

that is,

$$|\kappa(\theta)|^2 = \left[1 + \nu^2(\cos\theta - 1) + \frac{\mu_\alpha}{\Gamma(4-\alpha)}\sum_{m=-1}^{\infty} q_m\cos(m\theta)\right]^2 + [\nu\sin\theta]^2.$$

Let us consider $\theta = \pi$, which corresponds to the highest frequency resolvable on the mesh, namely frequency of wavelength $2\Delta x$. We have

$$\kappa(\pi) = 1 - 2\nu^2 + \frac{\mu_\alpha}{\Gamma(4-\alpha)} \sum_{m=-1}^{\infty} (-1)^m q_m.$$

Therefore $|\kappa(\pi)| \leq 1$ gives

$$0 \leq \nu^2 - \mu_\alpha \frac{1}{2\Gamma(4-\alpha)} \sum_{m=-1}^{\infty} (-1)^m q_m \leq 1,$$

and it is proved condition (42) is a necessary condition.

Let us now turn to the sufficient condition, that is, if we have condition (42) then $|\kappa(\theta)| \leq 1$, for all θ. We have that

$$\sum_{m=-1}^{\infty} q_m \cos(m\theta) = (q_{-1} + q_1)\cos(\theta) + q_0 + \sum_{m=2}^{\infty} q_m \cos(m\theta)$$

and since $q_{-1} + q_1 \geq 0$ and $q_m \geq 0,\ m \geq 2$ we get

$$\sum_{m=-1}^{\infty} q_m \cos(m\theta) \leq (q_{-1} + q_1)\cos\theta + q_0 + \sum_{m=2}^{\infty} q_m = (q_{-1} + q_1)(\cos\theta - 1).$$

From the fact that

$$\sum_{m=-1}^{\infty} q_m \cos(m\theta) \leq 0$$

we have

$$|\kappa(\theta)|^2 \leq \left[1 + \nu^2(\cos\theta - 1) + \frac{\mu_\alpha}{2\Gamma(4-\alpha)} \sum_{m=-1}^{\infty} q_m \cos(m\theta)\right]^2 + [\nu \sin\theta]^2.$$

Therefore

$$|\kappa(\theta)|^2 \leq \left[1 - \nu^2(1-\cos\theta) - \frac{\mu_\alpha}{2\Gamma(4-\alpha)}(q_{-1} + q_1)(1-\cos\theta)\right]^2 + [\nu\sin\theta]^2,$$

For

$$C = \nu^2 + \frac{\mu_\alpha}{2\Gamma(4-\alpha)}(q_{-1} + q_1),$$

it follows

$$\begin{aligned}|\kappa(\theta)|^2 &\leq [1 - C(1-\cos\theta)]^2 + [\nu\sin\theta]^2 \\ &= [1 - 2C\sin^2(\theta/2)]^2 + \nu^2\sin^2\theta \\ &= 1 - 4C\sin^2(\theta/2) + 4C^2\sin^4(\theta/2) + 2\nu^2\sin^2(\theta/2) - 2\nu^2\sin^4(\theta/2).\end{aligned}$$

Let $s = \sin(\theta/2)$. Then

$$\begin{aligned}|\kappa(\theta)|^2 &\leq 1 - [4C - 2\nu^2]s^2 + [4C^2 - 2\nu^2]s^4 \\ &\leq 1 - [4C - 4C^2]s^4,\end{aligned}$$

since $s^2 \leq s^4$. By hypothesis $C \leq 1$, since by (42),

$$\nu^2 + \frac{\mu_\alpha}{2\Gamma(4-\alpha)}(q_{-1} + q_1) \leq 1 + \frac{\mu_\alpha}{2\Gamma(4-\alpha)}\left[\sum_{m=2}^{\infty}(-1)^m q_m + q_0\right] \leq 1.$$

Note that

$$\sum_{m=2}^{\infty}(-1)^m q_m + q_0 \leq 0$$

since

$$\sum_{m=2}^{\infty}(-1)^m q_m + q_0 \leq \sum_{m=2}^{\infty} q_m + q_0 = -(q_1 + q_{-1}) - q_0 + q_0 = -(q_1 + q_{-1}).$$

Therefore, for $C \leq 1$, we have $|\kappa(\theta)|^2 \leq 1$, for all θ. □

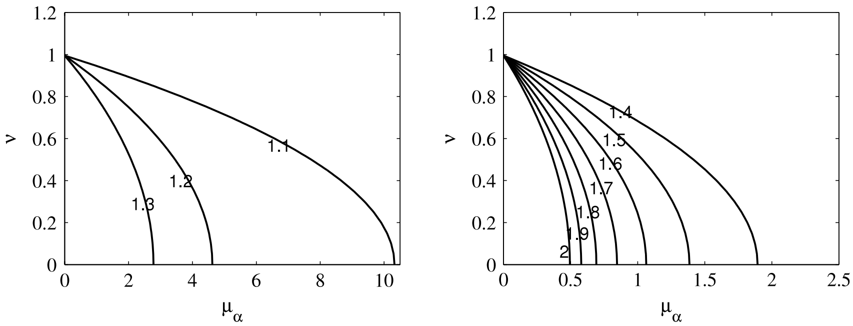

Fig. 1. Stability Fourier conditions (42). Plotted α changing. Necessary and sufficient conditions for $\beta = 0$. Necessary conditions for all $-1 \le \beta \le 1$.

The next theorem gives a necessary stability condition for all $-1 \le \beta \le 1$.

Theorem 3. *For $-1 \le \beta \le 1$, a necessary von Neumann stability condition for the numerical scheme is given by*

$$\nu^2 - \mu_\alpha \frac{1}{2\Gamma(4-\alpha)} \sum_{m=-1}^{\infty} (-1)^m q_m \le 1, \quad 1 < \alpha \le 2. \tag{43}$$

Proof. If we insert the mode $e^{ij\theta}$ into (41) we obtain the amplification factor

$$\kappa(\theta) = 1 - \frac{\nu}{2}(e^{i\theta} - e^{-i\theta}) + \frac{\nu^2}{2}(e^{i\theta} - 2 + e^{-i\theta}) + \frac{\mu_\alpha}{2\Gamma(4-\alpha)} \left((1+\beta) \sum_{m=-1}^{\infty} q_m e^{-im\theta} + (1-\beta) \sum_{m=-1}^{\infty} q_m e^{im\theta} \right)$$

that is,

$$\kappa(\theta) = 1 - i\nu \sin\theta + \nu^2(\cos\theta - 1) + \frac{\mu_\alpha}{\Gamma(4-\alpha)} \left(\sum_{m=-1}^{\infty} q_m \cos(m\theta) - i\beta \sum_{m=-1}^{\infty} q_m \sin(m\theta) \right).$$

For $\theta = \pi$ we have

$$\kappa(\pi) = 1 - 2\nu^2 + \frac{\mu_\alpha}{2\Gamma(4-\alpha)} \sum_{m=-1}^{\infty} q_m \cos(m\theta) \tag{44}$$

and therefore

$$\left| 1 - 2\nu^2 + \frac{\mu_\alpha}{\Gamma(4-\alpha)} \sum_{m=-1}^{\infty} q_m \cos(m\theta) \right| \le 1$$

and then we get the necessary condition (43). □

In Fig. 1, we plot the necessary and sufficient stability condition, for $\beta = 0$, and given by (42), which is also a necessary condition for $-1 \le \beta \le 1$.

In Figs. 2–5, we show the sufficient conditions for stability computed numerically through the calculus of $\|M\|_2 \le 1$. We observe that for $\alpha \ge 1.6$ the conditions are very similar to the condition (42) plotted in Fig. 1. Therefore the necessary condition (43) is very sharp for $\alpha \ge 1.6$ and $-1 \le \beta \le 1$. However, for $\alpha \le 1.6$ the stability condition (43) is not sharp when $\beta \ne 0$. Note that $\nu \le 1$ is a necessary condition for all the cases.

4. Numerical results

In this section we present some numerical tests to show the second order convergence of the numerical method by considering the l_∞ error, for an instant of time $t = n\Delta t$, given by

$$\max_j |u(x_j, t) - U_j^n|. \tag{45}$$

For the first example we assume $\beta = 1$ in Eq. (3), that is, we have the equation

$$\frac{\partial u}{\partial t} + V \frac{\partial u}{\partial x} = D \frac{\partial^\alpha u}{\partial x^\alpha} + p(x, t),$$

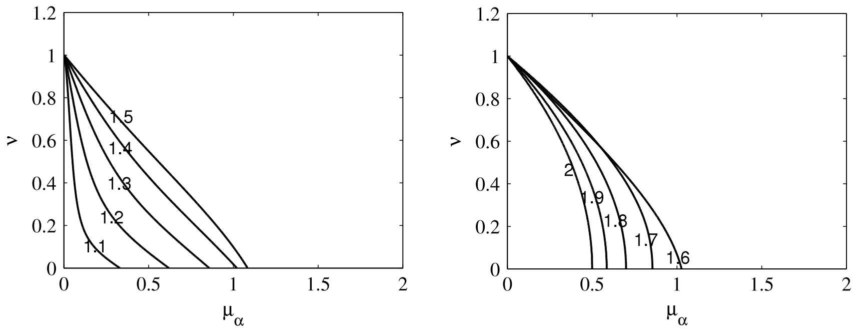

Fig. 2. Stability condition $\|M\|_2 \leq 1$. Plotted α changing for $\beta = -1$.

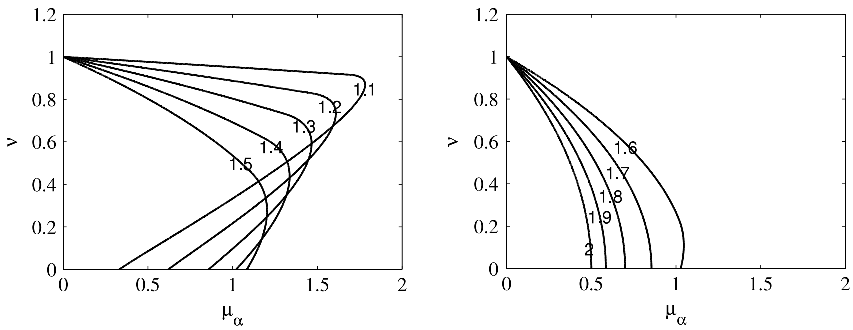

Fig. 3. Stability condition $\|M\|_2 \leq 1$. Plotted α changing for $\beta = 1$.

Fig. 4. Stability condition $\|M\|_2 \leq 1$. Plotted α changing for $\beta = -0.5$.

in the domain $0 \leq x \leq 1$. We assume the problem has initial condition $u(x, 0) = x^4$ and boundary conditions $u(0, t) = 0$, $u(1, t) = e^{-t}$. Let

$$V = 0.2, \qquad D = \frac{\Gamma(5-\alpha)}{24} \quad \text{and} \quad p(x, t) = e^{-t}x^3(4V - x - x^{1-\alpha}).$$

The exact solution is given by $u(x, t) = e^{-t}x^4$.

For this problem the scheme has the form

$$U_j^{n+1} = U_j^n - \nu\Delta_0 U_j^n + \frac{1}{2}\nu^2\delta^2 U_j^n + \mu_\alpha \delta_l^\alpha U_j^n + \Delta t \tilde{p}_j^n. \tag{46}$$

In Table 1 we present the l_∞ error, for the instant of time $t = 1$, which shows the numerical method has second order convergence.

The second example considers Eq. (3) for $\beta = 0$, that is, we have the equation

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} = \frac{D}{2}\left(\frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial^\alpha u}{\partial(-x)^\alpha}\right) + p(x, t),$$

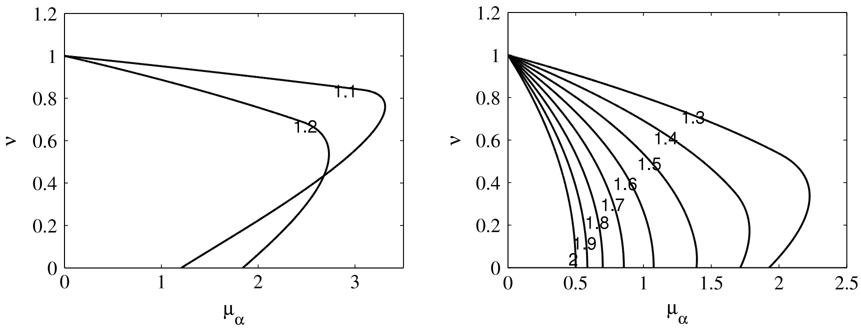

Fig. 5. Stability condition $\|M\|_2 \le 1$. Plotted α changing for $\beta = 0.5$.

Table 1
l_∞ error (45) at $t = 1$, for the numerical method (46).

Δx	$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$	$\alpha = 2$
1/10	0.4603×10^{-3}	0.3703×10^{-3}	0.4453×10^{-3}	0.7477×10^{-3}	0.1239×10^{-2}
1/100	0.5667×10^{-5}	0.4444×10^{-5}	0.3561×10^{-5}	0.4848×10^{-5}	0.1095×10^{-4}
Rate	1.90	1.92	2.09	2.19	2.05

Table 2
l_∞ error (45) at $t = 1$ for the numerical method (47).

Δx	$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$	$\alpha = 2$
2/10	0.5738×10^{-1}	0.6981×10^{-1}	0.7739×10^{-1}	0.8086×10^{-1}	0.7621×10^{-1}
2/100	0.5693×10^{-3}	0.5729×10^{-3}	0.6748×10^{-3}	0.7701×10^{-3}	0.7289×10^{-3}
Rate	2.0	2.09	2.06	2.02	2.02

in the domain $0 \le x \le 2$. We assume the initial condition is $u(x, 0) = 4x^2(2 - x)^2$ and the boundary conditions are $u(0, t) = 0$, $u(2, t) = 0$. Let

$$V = 0.05, \qquad D = \frac{\Gamma(5-\alpha)}{2}$$

and

$$p(x,t) = 4e^{-t}(-x^2(2-x)^2 + 4Vx(x^2 - 3x + 2) - x^{2-\alpha}A(x,\alpha) - (2-x)^{2-\alpha}B(x,\alpha))$$

where

$$A(x,\alpha) = 2\alpha(\alpha - 1) - 6\alpha(2 - x) + 6(2 - x)^2$$
$$B(x,\alpha) = 2\alpha(\alpha - 1) - 6\alpha x + 6x^2.$$

The exact solution is given by $u(x, t) = 4e^{-t}x^2(2 - x)^2$.

For this problem the numerical method has the form

$$U_j^{n+1} = U_j^n - \nu\Delta_0 U_j^n + \frac{1}{2}\nu^2\delta^2 U_j^n + \frac{1}{2}\mu_\alpha(\delta_l^\alpha U_j^n + \delta_r^\alpha U_j^n) + \Delta t\tilde{p}_j^n. \tag{47}$$

We observe both examples present second order convergence (see Table 2).

5. Final remarks

We have derived a second order numerical method for the fractional advection diffusion equation which is explicit. The convergence of the numerical method was analyzed through the consistency and the stability. For the stability analysis we have used Fourier analysis. Note that, in general, explicit numerical methods are more cost effective methods than the implicit methods. Additionally explicit methods are better tools for problems where advection plays an important role.

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