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An alternating direction implicit method for a second-order hyperbolic diffusion equation with convection ☆

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ABSTRACT

A numerical method is presented to solve a two-dimensional hyperbolic diffusion problem where is assumed that both convection and diffusion are responsible for flow motion. Since direct solutions based on implicit schemes for multidimensional problems are computationally inefficient, we apply an alternating direction method which is second order accurate in time and space. The stability of the alternating direction method is analyzed using the energy method. Numerical results are presented to illustrate the performance in different cases.

1. Introduction

Hyperbolic diffusion models have been widely discussed in literature since they overcome the unphysical property of infinite speed of propagation that is specific to parabolic models [10,14,15,24]. There are experimental evidences which prove that diffusive processes take place with finite velocity inside matter [4,12,13]. In some applications, this issue can be ignored but in many others it is necessary to take into account the wave nature of diffusive processes [22].

We consider a two dimensional hyperbolic transport equation that assumes that both convection and diffusion are responsible for flow motion, which can be seen as a more general telegrapher's equation [31]. Similar equations have been appearing in several works for different applications, such as, diffusive processes in the presence of a potential field [2,3,6], physical models with transport memory and nonlinear damping [21], hyperbolic models for convection–diffusion problems in computational fluid dynamics [11] and various heat transfer models [1,16,18,19,26,27]. Despite its great relevance in practical applications, the incorporation of a convection term has not been exhaustively studied.

In this work we derive a two-level alternating direction implicit (ADI) scheme to solve the two-dimensional problem. There are a great variety of applications of ADI methods based on the finite difference methods. When implicit methods are applied in one dimension, usually extensions to two dimensions require approaches such as the ADI, since they reduce the solution of a multidimensional problem to a set of independent one-dimensional problems and thus we obtain a more efficient method than the implicit schemes. For these reasons special attention has been given to these type of methods, when trying to solve multidimensional problems.

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Recently, some attention has been paid to the development and analysis of stable methods for numerical solutions for hyperbolic diffusive equations in two dimensions, including different ADI methods, such as, for the wave equation, two level ADI finite volume methods [30], two-level ADI Galerkin methods [33] and for the Sine–Gordon equation a three-level ADI method can be found in [5]. For the damped wave equation or telegraph equation we can find a vast literature, namely, three-level ADI methods combined with a Richardson extrapolation [7,8], three-level ADI finite difference methods [9,20,23] and two-level ADI finite difference methods [25,32].

However, we are not aware of any work that considers an ADI method for a diffusive dominated second order hyperbolic equation with convection, and as presented in this work, the inclusion of the convective term turns the stability analysis by the energy method much more intricate. Additionally, the less restrictive stability conditions obtained for the ADI approach are a great advantage for numerical implementation. Most of the numerical methods for the same type of equation are based in hybrid methods which involve applications of the Laplace techniques with control-volume formulation or finite-difference methods [2,3,18,19]. Stability conditions are not discussed theoretically in these works and this is one of the disadvantages of such approaches, since the stability region of the numerical methods needs to be found experimentally.

We consider the following two-dimensional hyperbolic equation, which includes diffusion and constant convection, defined in a rectangular domain $\Omega \subset \mathbb{R}^2$,

$$\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial t^2}+P\frac{\partial u}{\partial x}+Q\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2},\quad (x,y)\in\Omega,\quad t\in(0,T], \tag{1}$$

where P and Q are constants. It is also assumed that P and Q are less than one in absolute value, that is, less than the diffusion coefficient. If P and Q are larger than the diffusion coefficient, asymptotic analysis of exact solutions shows that the Cauchy problem of Eq. (1) can be unstable [31].

We consider the initial conditions given by

$$u(x,y,0)=u_0(x,y),\quad (x,y)\in\Omega, \tag{2}$$

$$\frac{\partial u}{\partial t}(x,y,0)=u_1(x,y),\quad (x,y)\in\Omega, \tag{3}$$

and Dirichlet boundary conditions

$$u(x,y,t)=f(x,y,t),\quad (x,y)\in\partial\Omega,\quad t\in(0,T]. \tag{4}$$

In Section 2, we start to describe the proposed method and also its truncation error. It consists first of deriving a scheme based in the Crank-Nicolson method and then, to overcome the computational inefficiency of an implicit scheme in two dimensions, we apply an ADI method. Then, in Section 3 we use the energy method for determining the stability of the finite difference method. In Section 4 we present some numerical experiments and we end with final remarks.

2. The second-order ADI numerical method

In this section we describe the numerical method applied to solve the problem (1)–(4) and also give its truncation error. Direct discretization of (1) leads to a finite difference scheme that is three-level in time. To avoid a three-level discretization scheme we introduce an auxiliary function, following the idea in [34]:

$$w=\frac{\partial u}{\partial t}+u \tag{5}$$

and change (1) into

$$\frac{\partial w}{\partial t}+P\frac{\partial u}{\partial x}+Q\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}. \tag{6}$$

Let us consider the mesh points in $\Omega=[a,b]\times[c,d]$ and $0\leqslant t\leqslant T$ given by $x_i=a+i\Delta x,\ i=0,\dots,N_x,\ y_j=c+j\Delta y,\ j=0,\dots,N_y$, where $\Delta x=(b-a)/N_x$ and $\Delta y=(d-c)/N_y$. Let $t_n=n\Delta t$, with Δt being the time increment and $n\Delta t\leqslant T$. We denote the approximate solutions to $u(x_i,y_j,t_n)$ and $w(x_i,y_j,t_n)$ by $U_{i,j}^n$ and $W_{i,j}^n$ respectively.

We define the following discretization operators. The first order forward and backward difference operators are given by

$$\delta_x^+U_{i,j}^n=\frac{U_{i+1,j}^n-U_{i,j}^n}{\Delta x},\quad \delta_y^+U_{i,j}^n=\frac{U_{i,j+1}^n-U_{i,j}^n}{\Delta y}$$

and

$$\delta_x^-U_{i,j}^n=\frac{U_{i,j}^n-U_{i-1,j}^n}{\Delta x},\quad \delta_y^-U_{i,j}^n=\frac{U_{i,j}^n-U_{i,j-1}^n}{\Delta y}.$$

The first order centered difference operators are given by

$$\delta_x U^n_{i,j} = \frac{1}{2}\left[\delta^+_x + \delta^-_x\right] U^n_{i,j} = \frac{U^n_{i+1,j} - U^n_{i-1,j}}{2\Delta x},$$

$$\delta_y U^n_{i,j} = \frac{1}{2}\left[\delta^+_y + \delta^-_y\right] U^n_{i,j} = \frac{U^n_{i,j+1} - U^n_{i,j-1}}{2\Delta y}$$

and the second order centered difference operators are defined by

$$\delta^2_x U^n_{i,j} = \frac{U^n_{i-1,j} - 2U^n_{i,j} + U^n_{i+1,j}}{\Delta x^2}, \quad \delta^2_y U^n_{i,j} = \frac{U^n_{i,j-1} - 2U^n_{i,j} + U^n_{i,j+1}}{\Delta y^2}.$$

We discretize Eqs. (5) and (6) using the Crank-Nicolson method:

$$W^{n+1}_{i,j} + W^n_{i,j} = U^{n+1}_{i,j} + U^n_{i,j} + \frac{2}{\Delta t}\left(U^{n+1}_{i,j} - U^n_{i,j}\right) \tag{7}$$

and

$$\begin{aligned}\frac{W^{n+1}_{i,j} - W^n_{i,j}}{\Delta t} = &-\frac{P}{2}\left[\frac{U^{n+1}_{i+1,j} - U^{n+1}_{i-1,j}}{2\Delta x} + \frac{U^n_{i+1,j} - U^n_{i-1,j}}{2\Delta x}\right] - \frac{Q}{2}\left[\frac{U^{n+1}_{i,j+1} - U^{n+1}_{i,j-1}}{2\Delta y} + \frac{U^n_{i,j+1} - U^n_{i,j-1}}{2\Delta y}\right] \\ &+\frac{1}{2}\left[\frac{U^{n+1}_{i-1,j} - 2U^{n+1}_{i,j} + U^{n+1}_{i+1,j}}{\Delta x^2} + \frac{U^n_{i-1,j} - 2U^n_{i,j} + U^n_{i+1,j}}{\Delta x^2}\right] \\ &+\frac{1}{2}\left[\frac{U^{n+1}_{i,j-1} - 2U^{n+1}_{i,j} + U^{n+1}_{i,j+1}}{\Delta y^2} + \frac{U^n_{i,j-1} - 2U^n_{i,j} + U^n_{i,j+1}}{\Delta y^2}\right].\end{aligned} \tag{8}$$

Using the discretization operators given above and by denoting the set of discretisations points $U^n = \{U^n_{i,j}\}$ and $W^n = \{W^n_{i,j}\}$ the numerical method (7), (8) can be written in the form

$$W^{n+1} + W^n = U^{n+1} + U^n + \frac{2}{\Delta t}\left(U^{n+1} - U^n\right) \tag{9}$$

and

$$W^{n+1} - W^n = -\frac{P\Delta t}{2}\delta_x\left(U^{n+1} + U^n\right) - \frac{Q\Delta t}{2}\delta_y\left(U^{n+1} + U^n\right) + \frac{\Delta t}{2}\delta^2_x\left(U^{n+1} + U^n\right) + \frac{\Delta t}{2}\delta^2_y\left(U^{n+1} + U^n\right). \tag{10}$$

For our two-dimensional problem, it is computationally inefficient to obtain a direct solution of this scheme. Therefore an ADI strategy is used to overcome this difficulty, since it is well known that ADI methods can change a multidimensional problem into a series of independent one-dimensional problems and have the advantage of less computational cost. The main idea of the ADI for the two dimensional problem is to split the computations in two steps. In the first step, we apply an implicit method in the x-direction and an explicit method in the y-direction, producing an intermediate solution for time. In the second step, we apply an implicit method in the y-direction and an explicit method in the x-direction. This is described bellow.

If we replace the exact solution in the numerical method (9), (10), then

$$w^{n+1} + w^n = u^{n+1} + u^n + \frac{2}{\Delta t}(u^{n+1} - u^n) + \mathcal{O}(\Delta t^2) \tag{11}$$

and

$$w^{n+1} - w^n = -\frac{P\Delta t}{2}\delta_x(u^{n+1} + u^n) - \frac{Q\Delta t}{2}\delta_y(u^{n+1} + u^n) + \frac{\Delta t}{2}\delta^2_x(u^{n+1} + u^n) + \frac{\Delta t}{2}\delta^2_y(u^{n+1} + u^n) + \mathcal{O}\left(\Delta t^2 + \Delta^2\right), \tag{12}$$

where $\Delta^2 = \Delta x^2 + \Delta y^2$ and the method is $\mathcal{O}\left(\Delta t^2 + \Delta^2\right)$ accurate. Replacing (11) in (12) we have

$$\left(1 + \frac{2}{\Delta t}\right)u^{n+1} + \left(1 - \frac{2}{\Delta t}\right)u^n = -\frac{\Delta t}{2}(P\delta_x + Q\delta_y)(u^{n+1} + u^n) + \frac{\Delta t}{2}\left(\delta^2_x + \delta^2_y\right)(u^{n+1} + u^n) + 2w^n + \mathcal{O}\left(\Delta t^2 + \Delta^2 + \Delta t\Delta^2\right).$$

Let us define the operators

$$L_P = \frac{\Delta t}{2}(P\delta_x - \delta^2_x) \quad \text{and} \quad L_Q = \frac{\Delta t}{2}(Q\delta_y - \delta^2_y). \tag{13}$$

Then, the previous equation can be re-written as

$$\left(\left(1 + \frac{2}{\Delta t}\right) + L_P + L_Q\right)u^{n+1} = \left(\left(1 + \frac{2}{\Delta t}\right) - L_P - L_Q\right)u^n \tag{14}$$

$$+ 2(w^n - u^n) + \mathcal{O}\left(\Delta t^2 + \Delta^2 + \Delta t\Delta^2\right). \tag{15}$$

The accuracy of the numerical method is $\mathcal{O}\left(\Delta t^2+\Delta^2\right)$, and therefore we can add to it any term of the same or higher order without changing the accuracy of the scheme. With this in mind, let us consider the term

$$\frac{\Delta t}{\Delta t+2}L_PL_Q(u^{n+1}-u^n). \tag{16}$$

Since

$$\begin{aligned}L_PL_Q(u^{n+1}-u^n)&=\frac{\Delta t^3}{4}\left(P\frac{\partial}{\partial x}-\frac{\partial^2}{\partial x^2}\right)\left(Q\frac{\partial}{\partial y}-\frac{\partial^2}{\partial y^2}\right)\frac{\partial u^n}{\partial t}+\mathcal{O}\left(\Delta t^3\Delta^2+\Delta t^4\right)\\&=\frac{\Delta t^3}{4}\left(P-\frac{\partial}{\partial x}\right)\left(Q-\frac{\partial}{\partial y}\right)\frac{\partial^3 u^n}{\partial x\partial y\partial t}+\mathcal{O}\left(\Delta t^3\Delta^2+\Delta t^4\right),\end{aligned}$$

the term (16) is $\mathcal{O}\left(\Delta t^3\Delta^2+\Delta t^4\right)$ and can be added to (15) without changing its order of accuracy. Then we obtain

$$\begin{aligned}\left(\left(1+\frac{2}{\Delta t}\right)+L_P\right)\left(1+\frac{\Delta t}{\Delta t+2}L_Q\right)u^{n+1}=&\left(\left(1+\frac{2}{\Delta t}\right)-L_P\right)\left(1-\frac{\Delta t}{\Delta t+2}L_Q\right)u^n+2(w^n-u^n)\\&+\mathcal{O}\left(\Delta t^2+\Delta^2+\Delta t\Delta^2+\Delta t^3\Delta^2\right).\end{aligned} \tag{17}$$

Therefore, we have the following numerical method

$$\left(\left(1+\frac{2}{\Delta t}\right)+L_P\right)\left(1+\frac{\Delta t}{\Delta t+2}L_Q\right)U^{n+1}=\left(\left(1+\frac{2}{\Delta t}\right)-L_P\right)\left(1-\frac{\Delta t}{\Delta t+2}L_Q\right)U^n+2(W^n-U^n). \tag{18}$$

To implement the previous method, and following the Peaceman-Rachford strategy we can split the previous equation in two, by introducing an intermediate variable $\widetilde{U}$, which represents a solution computed at an intermediate time. Therefore we obtain a type of Peaceman-Rachford ADI,

$$\left(\left(1+\frac{2}{\Delta t}\right)+L_P\right)\widetilde{U}=\left(1-\frac{\Delta t}{\Delta t+2}L_Q\right)U^n+\frac{\Delta t}{\Delta t+2}(W^n-U^n), \tag{19}$$

$$\left(1+\frac{\Delta t}{\Delta t+2}L_Q\right)U^{n+1}=\left(\left(1+\frac{2}{\Delta t}\right)-L_P\right)\widetilde{U}+\frac{\Delta t}{\Delta t+2}(W^n-U^n). \tag{20}$$

When solving the previous equations we need to pay attention to the boundary values of the intermediate variable $\widetilde{U}$, since in order to solve Eq. (19) for $\widetilde{U}_{i,j},\ i=1,\dots,N_x-1,\ j=1,\dots,N_y-1$, we need the values $\widetilde{U}_{0,j}$ and $\widetilde{U}_{N_x,j}$. Note that the boundary values $\widetilde{U}_{i,0}$ and $\widetilde{U}_{i,N_y}$ are not needed to solve Eq. (20).

A convenient way to find the boundary values $\widetilde{U}_{0,j}$ and $\widetilde{U}_{N_x,j}$ is to write Eqs. (19) and (20) in the equivalent form

$$\begin{aligned}\left(\left(1+\frac{2}{\Delta t}\right)+L_P\right)\widetilde{U}&=\left(1-\frac{\Delta t}{\Delta t+2}L_Q\right)U^n+\frac{\Delta t}{\Delta t+2}(W^n-U^n),\\\left(\left(1+\frac{2}{\Delta t}\right)-L_P\right)\widetilde{U}&=\left(1+\frac{\Delta t}{\Delta t+2}L_Q\right)U^{n+1}-\frac{\Delta t}{\Delta t+2}(W^n-U^n).\end{aligned}$$

By adding these equations we obtain, at each point (x_i,y_j),

$$\widetilde{U}_{i,j}=\frac{\Delta t}{2(\Delta t+2)}\left(1-\frac{\Delta t}{\Delta t+2}L_Q\right)U^n_{i,j}+\frac{\Delta t}{2(\Delta t+2)}\left(1+\frac{\Delta t}{\Delta t+2}L_Q\right)U^{n+1}_{i,j}.$$

This equation does not contain the operator L_P and we can now obtain easily the intermediate values $\widetilde{U}_{0,j}$ and $\widetilde{U}_{N_x,j}$, yielding

$$\begin{aligned}\widetilde{U}_{0,j}&=\frac{\Delta t}{2(\Delta t+2)}\left(1-\frac{\Delta t}{\Delta t+2}L_Q\right)U^n_{0,j}+\frac{\Delta t}{2(\Delta t+2)}\left(1+\frac{\Delta t}{\Delta t+2}L_Q\right)U^{n+1}_{0,j}\\&=\frac{\Delta t}{2(\Delta t+2)}\left(1-\frac{\Delta t}{\Delta t+2}L_Q\right)f(a,y_j,t_n)+\frac{\Delta t}{2(\Delta t+2)}\left(1+\frac{\Delta t}{\Delta t+2}L_Q\right)f(a,y_j,t_{n+1})\end{aligned} \tag{21}$$

and

$$\begin{aligned}\widetilde{U}_{N_x,j}&=\frac{\Delta t}{2(\Delta t+2)}\left(1-\frac{\Delta t}{\Delta t+2}L_Q\right)U^n_{N_x,j}+\frac{\Delta t}{2(\Delta t+2)}\left(1+\frac{\Delta t}{\Delta t+2}L_Q\right)U^{n+1}_{N_x,j}\\&=\frac{\Delta t}{2(\Delta t+2)}\left(1-\frac{\Delta t}{\Delta t+2}L_Q\right)f(b,y_j,t_n)+\frac{\Delta t}{2(\Delta t+2)}\left(1+\frac{\Delta t}{\Delta t+2}L_Q\right)f(b,y_j,t_{n+1}).\end{aligned} \tag{22}$$

3. Stability analysis

To prove the stability of the finite difference scheme (18) with respect to the initial values we use the discrete energy method. To this end, let us define the set of discrete values with homogeneous boundary conditions.

Assume that $\mathcal{G} = \{U|\ U = \{U_{i,j}\},\ \text{and}\ U_{0,j} = U_{N_x,j} = U_{i,0} = U_{i,N_y} = 0\}$. For $U, V \in \mathcal{G}$, we define the inner product and norm respectively as

$$(U,V) = \Delta x \Delta y \sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y-1} U_{i,j}V_{i,j}, \quad \|U\|^2 = (U,U). \tag{23}$$

Additionally, we also need to define other inner products that involve the first and second order discretization operators of $U, V \in \mathcal{G}$, since they will be used in the next lemmas and in the main theorem.

For $U, V \in \mathcal{G}$, let us define,

$$(\delta_x^+ U, \delta_x^+ V)_{*x} = \Delta x \Delta y \sum_{i=0}^{N_x-1}\sum_{j=1}^{N_y-1} \delta_x^+ U_{i,j}\delta_x^+ V_{i,j}, \quad \|\delta_x^+ U\|_{*x}^2 = (\delta_x^+ U, \delta_x^+ U)_{*x},$$

$$\left(\delta_y^+ U, \delta_y^+ V\right)_{*y} = \Delta x \Delta y \sum_{i=1}^{N_x-1}\sum_{j=0}^{N_y-1} \delta_y^+ U_{i,j}\delta_y^+ V_{i,j}, \quad \|\delta_y^+ U\|_{*y}^2 = \left(\delta_y^+ U, \delta_y^+ U\right)_{*y},$$

$$\left(\delta_x^+\delta_y^+ U, \delta_x^+\delta_y^+ V\right)_{*} = \Delta x \Delta y \sum_{i=0}^{N_x-1}\sum_{j=0}^{N_y-1} \delta_x^+\delta_y^+ U_{i,j}\delta_x^+\delta_y^+ V_{i,j}, \quad \|\delta_x^+\delta_y^+ U\|_{*}^2 = \left(\delta_x^+\delta_y^+ U, \delta_x^+\delta_y^+ U\right)_{*}.$$

Lemma 1. *For any $W \in \mathcal{G}$,*

$$\|\delta_x W\| \leqslant \|\delta_x^+ W\|_{*x}, \quad \|\delta_y W\| \leqslant \|\delta_y^+ W\|_{*y}, \quad \|\delta_x\delta_y W\| \leqslant \|\delta_x^+\delta_y^+ W\|_{*}.$$

Proof. We only prove the first inequality. The other inequalities follow in a similar way. We have

$$\begin{aligned}\|\delta_x W\|^2 &= \sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y-1}\Delta x\Delta y(\delta_x W_{i,j})^2 \leqslant \frac{1}{2}\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y-1}\Delta x\Delta y\left(\frac{W_{i+1,j}-W_{i,j}}{\Delta x}\right)^2 + \frac{1}{2}\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y-1}\Delta x\Delta y\left(\frac{W_{i,j}-W_{i-1,j}}{\Delta x}\right)^2 \\ &\leqslant \frac{1}{2}\sum_{i=0}^{N_x-1}\sum_{j=1}^{N_y-1}\Delta x\Delta y\left(\frac{W_{i+1,j}-W_{i,j}}{\Delta x}\right)^2 + \frac{1}{2}\sum_{i=0}^{N_x-2}\sum_{j=1}^{N_y-1}\Delta x\Delta y\left(\frac{W_{i+1,j}-W_{i,j}}{\Delta x}\right)^2 \leqslant \frac{1}{2}\|\delta_x^+ W\|_{*x}^2 + \frac{1}{2}\|\delta_x^+ W\|_{*x}^2. \quad \square\end{aligned}$$

The following lemma is the well-known property of summation by parts.

Lemma 2. *For any $U, V \in \mathcal{G}$,*

$$(\delta_x^2 U, V) = -(\delta_x^+ U, \delta_x^+ V)_{*x}, \quad \left(\delta_y^2 U, V\right) = -\left(\delta_y^+ U, \delta_y^+ V\right)_{*y}.$$

The next lemma can be seen, for instance in [25,28].

Lemma 3. *For any $U \in \mathcal{G}$, the following inequalities hold*

$$\|\delta_x^+ U\|_{*x}^2 \leqslant \frac{4}{\Delta x^2}\|U\|^2, \quad \|\delta_y^+ U\|_{*y}^2 \leqslant \frac{4}{\Delta y^2}\|U\|^2,$$

$$\|\delta_x^+\delta_y^+ U\|_{*}^2 \leqslant \frac{4}{\Delta y^2}\|\delta_x^+ U\|_{*x}^2, \quad \|\delta_x^+\delta_y^+ U\|_{*}^2 \leqslant \frac{4}{\Delta x^2}\|\delta_y^+ U\|_{*y}^2.$$

Before proving the main result, note that the ADI method can be written in the form

$$\left(1+\frac{2}{\Delta t}\right)U^{n+1} + \left(1-\frac{2}{\Delta t}\right)U^n - 2W^n + \frac{\Delta t}{\Delta t+2}L_P L_Q\left(U^{n+1}-U^n\right) = -(L_P+L_Q)\left(U^{n+1}+U^n\right). \tag{24}$$

Taking into account (7) we have

$$W^{n+1} - W^n + \frac{\Delta t}{\Delta t + 2} L_P L_Q \left(U^{n+1} - U^n\right) = -(L_P + L_Q)\left(U^{n+1} + U^n\right). \tag{25}$$

Theorem 4. *Suppose that $\{U^n_{i,j}, W^n_{i,j}\}$ and $\{V^n_{i,j}, Y^n_{i,j}\}$ are solutions of the finite difference scheme (25) which satisfy the boundary condition (4), and have different initial values $\{U^0_{i,j}, W^0_{i,j}\}$ and $\{V^0_{i,j}, Y^0_{i,j}\}$ respectively. Let $\omega^n_{i,j} = W^n_{i,j} - Y^n_{i,j},\ \epsilon^n_{i,j} = U^n_{i,j} - V^n_{i,j}$. For $\Delta t \leqslant 1$, such that, $\Delta t \leqslant c_p \Delta x,\ \Delta t \leqslant c_q \Delta y$, with constants c_p, c_q, then $\{\omega^n_{i,j}, \epsilon^n_{i,j}\}$ satisfy*

$$\|\omega^{n+1}\|^2 + \|\delta_x^+ \epsilon^{n+1}\|^2_{*x} + \|\delta_y^+ \epsilon^{n+1}\|^2_{*y} \leqslant (1 + C\Delta t)\left(\|\omega^n\|^2 + \|\delta_x^+ \epsilon^n\|^2_{*x} + \|\delta_y^+ \epsilon^n\|^2_{*y}\right), \tag{26}$$

where C denotes a constant independent of $\Delta x, \Delta y, \Delta t$.

Proof. We consider $\{U^n_{i,j}, W^n_{i,j}\}$ and $\{V^n_{i,j}, Y^n_{i,j}\}$ which are solutions of the finite difference scheme (25) and satisfy the same boundary condition (4). Therefore, for $\omega^n = \{\omega^n_{i,j}\}$ and $\epsilon^n = \{\epsilon^n_{i,j}\}$, we have $\{\omega^n, \epsilon^n\} \in \mathcal{G}$, that is, these discrete points at the boundary are zero. From (25) they also satisfy

$$\omega^{n+1} - \omega^n + \frac{\Delta t}{\Delta t + 2} L_P L_Q (\epsilon^{n+1} - \epsilon^n) = -(L_P + L_Q)(\epsilon^{n+1} + \epsilon^n). \tag{27}$$

Multiplying both sides of (27) by $\omega^{n+1} + \omega^n$ with respect to the inner product (23) we obtain, using (13),

$$\begin{aligned}
&\|\omega^{n+1}\|^2 - \|\omega^n\|^2 + \frac{\Delta t^3}{4(\Delta t + 2)}\left(\left(PQ\delta_x\delta_y - P\delta_x\delta_y^2 - Q\delta_x^2\delta_y + \delta_x^2\delta_y^2\right)(\epsilon^{n+1} - \epsilon^n), \omega^{n+1} + \omega^n\right) \\
&\quad + \frac{\Delta t}{2}\left(\left(P\delta_x - \delta_x^2 + Q\delta_y - \delta_y^2\right)(\epsilon^{n+1} + \epsilon^n), \omega^{n+1} + \omega^n\right) = 0.
\end{aligned} \tag{28}$$

By (9) and summation by parts we have

$$\left(\delta_x^2\delta_y^2(\epsilon^{n+1} - \epsilon^n), \omega^{n+1} + \omega^n\right) = \|\delta_x^+\delta_y^+\epsilon^{n+1}\|^2_* - \|\delta_x^+\delta_y^+\epsilon^n\|^2_* + \frac{2}{\Delta t}\|\delta_x^+\delta_y^+(\epsilon^{n+1} - \epsilon^n)\|^2_*,$$

$$(\delta_x^2(\epsilon^{n+1} + \epsilon^n), \omega^{n+1} + \omega^n) = -\|\delta_x^+(\epsilon^{n+1} + \epsilon^n)\|^2_{*x} - \frac{2}{\Delta t}\left(\|\delta_x^+\epsilon^{n+1}\|^2_{*x} - \|\delta_x^+\epsilon^n\|^2_{*x}\right)$$

and

$$\left(\delta_y^2(\epsilon^{n+1} + \epsilon^n), \omega^{n+1} + \omega^n\right) = -\|\delta_y^+(\epsilon^{n+1} + \epsilon^n)\|^2_{*y} - \frac{2}{\Delta t}\left(\|\delta_y^+\epsilon^{n+1}\|^2_{*y} - \|\delta_y^+\epsilon^n\|^2_{*y}\right).$$

We can re-write (28) as

$$\begin{aligned}
&\|\omega^{n+1}\|^2 + \frac{\Delta t^3}{4(\Delta t + 2)}\|\delta_x^+\delta_y^+\epsilon^{n+1}\|^2_* + \|\delta_x^+\epsilon^{n+1}\|^2_{*x} + \|\delta_y^+\epsilon^{n+1}\|^2_{*y} \\
&\quad = \|\omega^n\|^2 + \frac{\Delta t^3}{4(\Delta t + 2)}\|\delta_x^+\delta_y^+\epsilon^n\|^2_* + \|\delta_x^+\epsilon^n\|^2_{*x} + \|\delta_y^+\epsilon^n\|^2_{*y} - \frac{\Delta t}{2}\left(\|\delta_x^+(\epsilon^{n+1} + \epsilon^n)\|^2_{*x} + \|\delta_y^+(\epsilon^{n+1} + \epsilon^n)\|^2_{*y}\right) \\
&\qquad - \frac{\Delta t^2}{2(\Delta t + 2)}\|\delta_x^+\delta_y^+(\epsilon^{n+1} - \epsilon^n)\|^2_* - \frac{\Delta t}{2}\left((P\delta_x + Q\delta_y)(\epsilon^{n+1} + \epsilon^n), \omega^{n+1} + \omega^n\right) \\
&\qquad - \frac{\Delta t^3}{4(\Delta t + 2)}\left(\left(PQ\delta_x\delta_y - P\delta_x\delta_y^2 - Q\delta_x^2\delta_y\right)(\epsilon^{n+1} - \epsilon^n), \omega^{n+1} + \omega^n\right).
\end{aligned} \tag{29}$$

Let us now discuss the terms with P or Q. We first consider the terms

$$-(P\delta_x(\epsilon^{n+1} + \epsilon^n), \omega^{n+1} + \omega^n) \quad \text{and} \quad -\left(Q\delta_y(\epsilon^{n+1} + \epsilon^n), \omega^{n+1} + \omega^n\right).$$

Using the Cauchy–Schwarz inequality, Lemma 1 and the inequality $ab \leqslant \eta a^2 + b^2/4\eta$, for $\eta > 0$, we have

$$\begin{aligned}
-(P\delta_x(\epsilon^{n+1} + \epsilon^n), \omega^{n+1} + \omega^n) &\leqslant \|P\delta_x(\epsilon^{n+1} + \epsilon^n)\|\|\omega^{n+1} + \omega^n\| \leqslant \||P|\delta_x^+(\epsilon^{n+1} + \epsilon^n)\|_{*x}\|\omega^{n+1} + \omega^n\| \\
&\leqslant \eta_1|P|^2\|\delta_x^+(\epsilon^{n+1} + \epsilon^n)\|^2_{*x} + \frac{1}{4\eta_1}\|\omega^{n+1} + \omega^n\|^2.
\end{aligned}$$

Using the inequality $(a+b)^2 \leqslant 2a^2 + 2b^2$, we conclude that

$$-(P\delta_x(\epsilon^{n+1} + \epsilon^n), \omega^{n+1} + \omega^n) \leqslant \eta_1|P|^2\|\delta_x^+(\epsilon^{n+1} + \epsilon^n)\|^2_{*x} + \frac{1}{2\eta_1}\left(\|\omega^{n+1}\|^2 + \|\omega^n\|^2\right). \tag{30}$$

Similarly

$$-\left(Q\delta_y(\epsilon^{n+1}+\epsilon^n),\omega^{n+1}+\omega^n\right) \leqslant \eta_2|Q|^2\|\delta_y^+(\epsilon^{n+1}+\epsilon^n)\|_{*y}^2 + \frac{1}{2\eta_2}\left(\|\omega^{n+1}\|^2+\|\omega^n\|^2\right). \tag{31}$$

Let us now consider the term

$$-\left(PQ\delta_x\delta_y(\epsilon^{n+1}-\epsilon^n),\omega^{n+1}+\omega^n\right).$$

Using the Cauchy–Schwarz inequality, Lemma 1 and the inequalities used previously we have

$$-\left(PQ\delta_x\delta_y(\epsilon^{n+1}-\epsilon^n),\omega^{n+1}+\omega^n\right) \leqslant \eta_3|PQ|^2\|\delta_x^+\delta_y^+(\epsilon^{n+1}-\epsilon^n)\|_*^2 + \frac{1}{2\eta_3}\left(\|\omega^{n+1}\|^2+\|\omega^n\|^2\right). \tag{32}$$

Finally, we consider the terms

$$\left(P\delta_x\delta_y^2(\epsilon^{n+1}-\epsilon^n),\omega^{n+1}+\omega^n\right) \quad \text{and} \quad \left(Q\delta_y\delta_x^2(\epsilon^{n+1}-\epsilon^n),\omega^{n+1}+\omega^n\right).$$

Using summation by parts, Lemma 1 and the inequalities used previously, we obtain

$$\left(P\delta_x\delta_y^2(\epsilon^{n+1}-\epsilon^n),\omega^{n+1}+\omega^n\right) \leqslant \eta_4|P|^2\|\delta_x^+\delta_y^+(\epsilon^{n+1}-\epsilon^n)\|_*^2 + \frac{1}{4\eta_4}\|\delta_y^+(\omega^{n+1}+\omega^n)\|_{*y}^2.$$

Using Lemma 3 and the inequalities,

$$\left(P\delta_x\delta_y^2(\epsilon^{n+1}-\epsilon^n),\omega^{n+1}+\omega^n\right) \leqslant \eta_4|P|^2\|\delta_x^+\delta_y^+(\epsilon^{n+1}-\epsilon^n)\|_*^2 + \frac{2}{\eta_4\Delta y^2}\left(\|\omega^{n+1}\|^2+\|\omega^n\|^2\right). \tag{33}$$

Similarly

$$\left(Q\delta_y\delta_x^2(\epsilon^{n+1}-\epsilon^n),\omega^{n+1}+\omega^n\right) \leqslant \eta_5|Q|^2\|\delta_x^+\delta_y^+(\epsilon^{n+1}-\epsilon^n)\|_*^2 + \frac{2}{\eta_5\Delta x^2}\left(\|\omega^{n+1}\|^2+\|\omega^n\|^2\right). \tag{34}$$

From (29) and the inequalities (30)–(34), we obtain

$$\begin{aligned}
&\|\omega^{n+1}\|^2 + \frac{\Delta t^3}{4(\Delta t+2)}\|\delta_x^+\delta_y^+\epsilon^{n+1}\|_*^2 + \|\delta_x^+\epsilon^{n+1}\|_{*x}^2 + \|\delta_y^+\epsilon^{n+1}\|_{*y}^2 \\
&\quad \leqslant \|\omega^n\|^2 + \frac{\Delta t^3}{4(\Delta t+2)}\|\delta_x^+\delta_y^+\epsilon^n\|_*^2 + \|\delta_x^+\epsilon^n\|_{*x}^2 + \|\delta_y^+\epsilon^n\|_{*y}^2 - \frac{\Delta t}{2}\left(\|\delta_x^+(\epsilon^{n+1}+\epsilon^n)\|_{*x}^2 + \|\delta_y^+(\epsilon^{n+1}+\epsilon^n)\|_{*y}^2\right) \\
&\qquad - \frac{\Delta t^2}{2(\Delta t+2)}\|\delta_x^+\delta_y^+(\epsilon^{n+1}-\epsilon^n)\|_*^2 \\
&\qquad + \frac{\Delta t}{2}\left(\eta_1|P|^2\|\delta_x^+(\epsilon^{n+1}+\epsilon^n)\|_{*x}^2 + \eta_2|Q|^2\|\delta_y^+(\epsilon^{n+1}+\epsilon^n)\|_{*y}^2 + \frac{1}{2}(\eta_1^{-1}+\eta_2^{-1})\left(\|\omega^{n+1}\|^2+\|\omega^n\|^2\right)\right) \\
&\qquad + \frac{\Delta t^3}{4(\Delta t+2)}\left(\left(|PQ|^2\eta_3+|P|^2\eta_4+|Q|^2\eta_5\right)\|\delta_x^+\delta_y^+(\epsilon^{n+1}-\epsilon^n)\|_*^2 + \frac{1}{2}(\eta_3^{-1}+4\eta_4^{-1}\Delta y^{-2}+4\eta_5^{-1}\Delta x^{-2})\left(\|\omega^{n+1}\|^2+\|\omega^n\|^2\right)\right).
\end{aligned} \tag{35}$$

Reorganizing the terms and using the conditions $\Delta t \leqslant c_p\Delta x$ and $\Delta t \leqslant c_q\Delta y$, we obtain

$$\begin{aligned}
&A\|\omega^{n+1}\|^2 + \frac{\Delta t^3}{4(\Delta t+2)}\|\delta_x^+\delta_y^+\epsilon^{n+1}\|_*^2 + \|\delta_x^+\epsilon^{n+1}\|_{*x}^2 + \|\delta_y^+\epsilon^{n+1}\|_{*y}^2 \\
&\quad \leqslant B\|\omega^n\|^2 + \frac{\Delta t^3}{4(\Delta t+2)}\|\delta_x^+\delta_y^+\epsilon^n\|_*^2 + \|\delta_x^+\epsilon^n\|_{*x}^2 + \|\delta_y^+\epsilon^n\|_{*y}^2 + \frac{\Delta t}{2}\|\delta_x^+(\epsilon^{n+1}+\epsilon^n)\|_{*x}^2(\eta_1|P|^2-1) \\
&\qquad + \frac{\Delta t}{2}\|\delta_y^+(\epsilon^{n+1}+\epsilon^n)\|_{*y}^2(\eta_2|Q|^2-1) + \frac{\Delta t^2}{4(\Delta t+2)}\|\delta_x^+\delta_y^+(\epsilon^{n+1}-\epsilon^n)\|_*^2\left(\Delta t\left(|PQ|^2\eta_3+|P|^2\eta_4+|Q|^2\eta_5\right)-2\right),
\end{aligned} \tag{36}$$

where A and B are given by,

$$A = 1 - \frac{\Delta t}{2}\left(\frac{\eta_1^{-1}+\eta_2^{-1}}{2} + \frac{c_p^2\eta_5^{-1}+c_q^2\eta_4^{-1}}{\Delta t+2} + \frac{\Delta t^2\eta_3^{-1}}{4(\Delta t+2)}\right) \tag{37}$$

and

$$B = 1 + \frac{\Delta t}{2}\left(\frac{\eta_1^{-1}+\eta_2^{-1}}{2} + \frac{c_p^2\eta_5^{-1}+c_q^2\eta_4^{-1}}{\Delta t+2} + \frac{\Delta t^2\eta_3^{-1}}{4(\Delta t+2)}\right). \tag{38}$$

Let us choose $\eta_1 = \eta_2 = \eta_3 = \eta_4 = \eta_5 = \eta$ with

$$\eta \leqslant \min\left\{\frac{1}{P^2}, \frac{1}{Q^2}, \frac{2}{(PQ)^2 + P^2 + Q^2}\right\}. \tag{39}$$

Then, from (36) we have

$$\begin{aligned} &A\|\omega^{n+1}\|^2 + \frac{\Delta t^3}{4(\Delta t + 2)}\|\delta_x^+\delta_y^+\epsilon^{n+1}\|_*^2 + \|\delta_x^+\epsilon^{n+1}\|_{*x}^2 + \|\delta_y^+\epsilon^{n+1}\|_{*y}^2 \\ &\quad \leqslant B\|\omega^n\|^2 + \frac{\Delta t^3}{4(\Delta t + 2)}\|\delta_x^+\delta_y^+\epsilon^n\|_*^2 + \|\delta_x^+\epsilon^n\|_{*x}^2 + \|\delta_y^+\epsilon^n\|_{*y}^2, \end{aligned} \tag{40}$$

with

$$A = 1 - \frac{\Delta t}{2}\left(\eta^{-1} + \eta^{-1}\frac{c_p^2 + c_q^2}{\Delta t + 2} + \frac{\Delta t^2\eta^{-1}}{4(\Delta t + 2)}\right) \tag{41}$$

and

$$B = 1 + \frac{\Delta t}{2}\left(\eta^{-1} + \eta^{-1}\frac{c_p^2 + c_q^2}{\Delta t + 2} + \frac{\Delta t^2\eta^{-1}}{4(\Delta t + 2)}\right). \tag{42}$$

Using Lemma 3 and $\Delta t \leqslant c_q\Delta y$, from (40) we obtain

$$A\|\omega^{n+1}\|^2 + \|\delta_x^+\epsilon^{n+1}\|_{*x}^2 + \|\delta_y^+\epsilon^{n+1}\|_{*y}^2 \leqslant B\|\omega^n\|^2 + \left(1 + \frac{c_q^2\Delta t}{\Delta t + 2}\right)\|\delta_x^+\epsilon^n\|_{*x}^2 + \|\delta_y^+\epsilon^n\|_{*y}^2. \tag{43}$$

If we choose η such that

$$\eta > \frac{1}{2}(1 + M), \quad \text{for } M = \max\left\{\frac{c_p^2 + c_q^2}{2}, \frac{4(c_p^2 + c_q^2) + 1}{12}\right\},$$

we can easily check that $0 < A \leqslant 1$. Additionally if we choose η such that

$$\eta \leqslant \frac{1}{c_q^2} + \frac{c_p^2 + c_q^2}{2c_q^2}$$

we have $B \geqslant 1 + c_q^2\Delta t/(\Delta t + 2)$. Therefore by choosing η such that

$$\frac{1}{2}(1 + M) < \eta \leqslant \min\left\{\frac{1}{P^2}, \frac{1}{Q^2}, \frac{2}{(PQ)^2 + P^2 + Q^2}, \frac{2 + c_p^2 + c_q^2}{2c_q^2}\right\}, \tag{44}$$

it follows

$$A\left(\|\omega^{n+1}\|^2 + \|\delta_x^+\epsilon^{n+1}\|_{*x}^2 + \|\delta_y^+\epsilon^{n+1}\|_{*y}^2\right) \leqslant B\left(\|\omega^n\|^2 + \|\delta_x^+\epsilon^n\|_{*x}^2 + \|\delta_y^+\epsilon^n\|_{*y}^2\right). \tag{45}$$

Consequently, by noting that

$$\frac{B}{A} = 1 + \Delta t\frac{\eta^{-1} + \eta^{-1}\frac{c_p^2 + c_q^2}{\Delta t + 2} + \frac{\Delta t^2\eta^{-1}}{4(\Delta t + 2)}}{1 - \frac{\Delta t}{2}\left(\eta^{-1} + \eta^{-1}\frac{c_p^2 + c_q^2}{\Delta t + 2} + \frac{\Delta t^2\eta^{-1}}{4(\Delta t + 2)}\right)} \leqslant 1 + C\Delta t,$$

where C denotes a constant independent of $\Delta x, \Delta y, \Delta t$, we obtain the main result. □

From the previous theorem we get the following result.

Corollary 5. *Suppose that $\{U_{i,j}^n, W_{i,j}^n\}$ and $\{V_{i,j}^n, Y_{i,j}^n\}$ are solutions of the finite difference scheme (25) which satisfy the boundary condition (4), and have different initial values $\{U_{i,j}^0, W_{i,j}^0\}$ and $\{V_{i,j}^0, Y_{i,j}^0\}$ respectively. Let $\omega_{i,j}^n = W_{i,j}^n - Y_{i,j}^n$, $\epsilon_{i,j}^n = U_{i,j}^n - V_{i,j}^n$. For $\Delta t \leqslant 1$, such that, $\Delta t \leqslant c_p\Delta x$, $\Delta t \leqslant c_q\Delta y$, with constants c_p, c_q, then $\{\omega_{i,j}^n, \epsilon_{i,j}^n\}$ satisfy*

$$||\omega^n||^2 + ||\delta_x^+\epsilon^n||_{*x}^2 + ||\delta_y^+\epsilon^n||_{*y}^2 \leqslant K\left(||\omega^0||^2 + ||\delta_x^+\epsilon^0||_{*x}^2 + ||\delta_y^+\epsilon^0||_{*y}^2\right), \tag{46}$$

where K denotes a constant independent of $\Delta x, \Delta y, \Delta t$.

Remark 1. The previous results require that the maximum time step size is directly proportional to the space mesh sizes. Usually optimal results are obtained when time-step and space-steps are comparable and therefore this is a natural condition. Similar conditions can be seen in literature for ADI numerical methods for hyperbolic problems, that usually do not include the first order derivatives in space [7,8,17,29,33].

Remark 2. The choice of constants c_p and c_q mentioned in the previous theorem can depend on the values of P and Q as can be concluded by observing the condition (44). For the case when $P = 0$ or $Q = 0$, this condition can be easily adjusted and do not depend on P or Q respectively. A practical choice could be to consider $c_p^2 + c_q^2 \leqslant 2/3$, for all P, Q.

4. Numerical experiments

In this section we present numerical tests for two dimensional problems. We compare the numerical results with exact solutions, and we also illustrate the behavior of some of the solutions. Let

$$\epsilon_{i,j} = u_{i,j} - U_{i,j} \quad \omega_{i,j} = w_{i,j} - W_{i,j}, \tag{47}$$

where u is the exact solution, w is defined by (5) and U and W are the approximate solutions, respectively. To measure the error and the convergence rate we consider the norms of $||\epsilon||$ and $||\omega||$ defined by (23).

We present two problems for which we are able to determine the exact solution in order to compute the errors and the convergence rate. The third problem shows how the solution behaves, when we have an initial Gaussian condition and homogeneous boundary conditions. We start with a problem with $P = Q = 0$ and then we consider a more general problem.

Problem 1. Consider the problem

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad x \in]0, \sqrt{8}\pi[, \quad y \in]0, \sqrt{8}\pi[, \quad t > 0,$$

with initial conditions

$$u(x,y,0) = \sin\left(\frac{x}{\sqrt{8}}\right) \sin\left(\frac{y}{\sqrt{8}}\right), \quad (x,y) \in \left[0, \sqrt{8}\pi\right] \times \left[0, \sqrt{8}\pi\right], \tag{48}$$

$$\frac{\partial u}{\partial t}(x,y,0) = -\frac{1}{2} \sin\left(\frac{x}{\sqrt{8}}\right) \sin\left(\frac{y}{\sqrt{8}}\right), \quad (x,y) \in \left[0, \sqrt{8}\pi\right] \times \left[0, \sqrt{8}\pi\right], \tag{49}$$

and boundary conditions

$$u(0,y,t) = 0, \quad u\left(\sqrt{8}\pi, y, t\right) = 0, \tag{50}$$

$$u(x,0,t) = 0, \quad u\left(x, \sqrt{8}\pi, t\right) = 0. \tag{51}$$

The exact solution is given by

$$u(x,y,t) = e^{-t/2} \sin\left(\frac{x}{\sqrt{8}}\right) \sin\left(\frac{y}{\sqrt{8}}\right). \tag{52}$$

In Table 1, we present the errors $||\epsilon||$ and $||\omega||$ defined by (23), at the instant of time $t = 1$. We consider $\Delta y = \Delta x$ and $\Delta t = \Delta x$. We observe the convergence rate is second order as expected.

Problem 2. Let us now consider the problem

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} + P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad (x,y) \in (0,1) \times (0,1), \quad t > 0,$$

with initial conditions

$$u(x,y,0) = e^{Px/2+Qy/2} \sinh(x\sqrt{(4+P^2)/2}) \sinh(y\sqrt{(4+Q^2)/2}),$$

$$\frac{\partial u}{\partial t}(x,y,0) = -\frac{1+\sqrt{17+P^2+Q^2}}{2} e^{Px/2+Qy/2} \times \sinh(x\sqrt{(4+P^2)/2}) \sinh(y\sqrt{(4+Q^2)/2}),$$

Table 1
Errors $||\epsilon||$ and $||\omega||$ defined by (23) for $t = 1$, $0 \leqslant x, y \leqslant \sqrt{8}\pi$, $\Delta y = \Delta x$ and $\Delta t = \Delta x$.

Δx	Error $\|\|\epsilon\|\|$	Rate	Error $\|\|\omega\|\|$	Rate
$\sqrt{8}\pi/50$	7.361e − 04		1.468e − 04	
$\sqrt{8}\pi/100$	1.916e − 04	1.94	3.924e − 05	1.90
$\sqrt{8}\pi/300$	2.134e − 05	2.00	4.516e − 06	1.97
$\sqrt{8}\pi/500$	7.737e − 06	1.99	1.643e − 06	1.98

Table 2
Errors $||\epsilon||$ and $||\omega||$ defined by (23) for $P = 0.5,\ Q = 0.4,\ t = 1,\ 0 \leqslant x, y \leqslant 1,\ \Delta y = \Delta x$ and $\Delta t = \Delta x$.

Δx	Error $\|\|\epsilon\|\|$	Rate	Error $\|\|\omega\|\|$	Rate
1/50	5.653e – 05		2.689e – 04	
1/100	1.413e – 05	2.00	6.832e – 05	1.98
1/300	1.570e – 06	2.00	7.728e – 06	1.98
1/500	5.653e – 07	2.00	2.799e – 06	1.99

and boundary conditions

$$u(0,y,t) = u(x,0,t) = 0,$$

$$u(1,y,t) = e^{-(1+\sqrt{17+P^2+Q^2})t/2} e^{\frac{P}{2}+\frac{Qy}{2}} \sinh(\sqrt{(4+P^2)/2}) \sinh(y\sqrt{(4+Q^2)/2}),$$

$$u(x,1,t) = e^{-(1+\sqrt{17+P^2+Q^2})t/2} e^{\frac{Px}{2}+\frac{Q}{2}} \sinh(x\sqrt{(4+P^2)/2}) \sinh(\sqrt{(4+Q^2)/2}).$$

The exact solution is given by

$$u(x,y,t) = e^{-(1+\sqrt{17+P^2+Q^2})t/2} e^{\frac{Px}{2}+\frac{Qy}{2}} \sinh(x\sqrt{(4+P^2)/2}) \sinh(y\sqrt{(4+Q^2)/2}).$$

In the next tables we present the errors and convergence rates for different velocity fields, such as, in Table 2, for $P = 0.5,\ Q = 0.4$, in Table 3, for $P = 0.5,\ Q = -0.4$, and in Table 4, for $P = 0,\ Q = 0.5$. For all the cases we observe the convergence rate is second order.

Table 3
Errors $||\epsilon||$ and $||\omega||$ defined by (23) for $P = 0.5,\ Q = -0.4,\ t = 1,\ 0 \leqslant x, y \leqslant 1,\ \Delta y = \Delta x$ and $\Delta t = \Delta x$.

Δx	Error $\|\|\epsilon\|\|$	Rate	Error $\|\|\omega\|\|$	Rate
1/50	4.861e – 05		2.298e – 04	
1/100	1.216e – 05	2.00	5.888e – 05	1.96
1/300	1.352e – 06	2.00	6.702e – 06	1.98
1/500	4.866e – 07	2.00	2.431e – 06	1.99

Table 4
Errors $||\epsilon||$ and $||\omega||$ defined by (23) for $P = 0,\ Q = 0.5,\ t = 1,\ 0 \leqslant x, y \leqslant 1$, $\Delta y = \Delta x$ and $\Delta t = \Delta x$.

Δx	Error $\|\|\epsilon\|\|$	Rate	Error $\|\|\omega\|\|$	Rate
1/50	5.072e – 05		2.384e – 04	
1/100	1.268e – 05	2.00	6.080e – 05	1.97
1/300	1.409e – 06	2.00	6.897e – 06	1.98
1/500	5.074e – 07	2.00	2.500e – 06	1.99

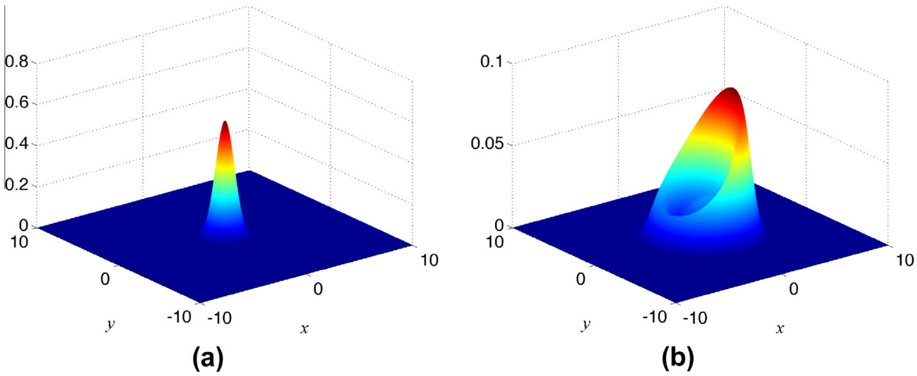

Fig. 1. (a) Initial condition; (b) approximate solution for $t = 3$ and $P = 0.5,\ Q = 0$. Computed with $\Delta t = \Delta x = \Delta y = 0.02$.

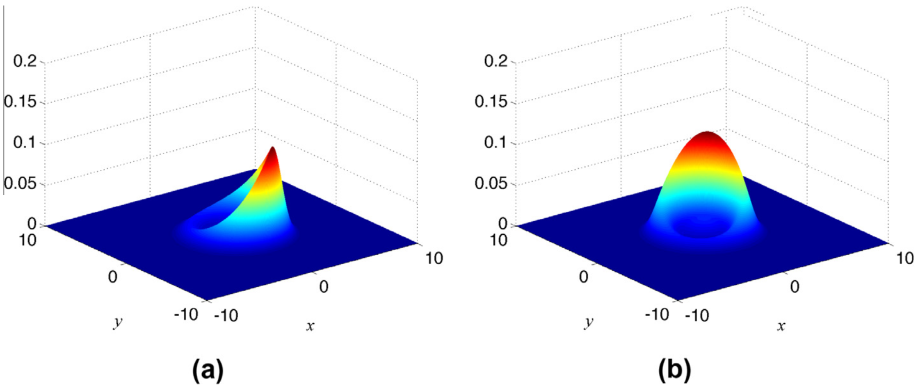

Fig. 2. Approximate solution for $t = 3$ computed with $\Delta t = \Delta x = \Delta y = 0.02$: (a) $P = 0.5,\ Q = -0.5$; (b) $P = 0.5,\ Q = 0.5$.

For this case the boundaries on the x direction are not zero, namely at $x = 1$, and therefore it indicates the formulation at the boundary values for the intermediate point, defined in (21), (22) does not affect the accuracy of the method.

Note also that for the tests presented, the step sizes satisfy the stability restrictions of Theorem 4. This reassures the restrictions are very reasonable, since we do not need to impose very small time steps.

Problem 3. This example gives an insight on the physical behavior of the solutions associated with this type of problems. We consider

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} + P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad (x, y) \in \mathbb{R}^2, \quad t > 0,$$

with initial conditions

$$u(x, y, 0) = \frac{1}{\sqrt{\pi}} e^{-(x^2+y^2)}, \quad \frac{\partial u}{\partial t}(x, y, 0) = 0,$$

and boundary conditions

$$\lim_{|x|\to\infty} u(x, y, t) = 0, \quad \lim_{|y|\to\infty} u(x, y, t) = 0.$$

Since we consider an infinite physical domain and the computational domain needs to be finite, for numerical implementation purposes we assume the computational domain is large enough, that is, until the solution is zero at those numerical boundaries (or very close to zero), in order to avoid the influence of the numerical boundaries in the computation and consequently on the accuracy of the numerical method. The examples presented are for the instant of time $t = 3$ and we have considered the spatial computational domain $[-10, 10] \times [-10, 10]$.

In Figs. 1,2 we display those approximate solutions and observe how the solution changes with the direction of the velocity field.

5. Final remarks

We have derived a second order accurate ADI finite difference method to solve a two dimensional hyperbolic equation, with Dirichlet boundary conditions. The stability of the method has been shown by the discrete energy method in Theorem 4. Although it presents sufficient conditions for stability, they are not very restrictive in practice, allowing the choice of large time steps. Numerical results demonstrate the second order accuracy and efficiency of the numerical method and the results are in agreement with the theoretical analysis presented.

References

[1] Y.M. Ali, L.C. Zhang, Relativistic moving heat source, Int. J. Heat Mass Transfer 48 (2005) 2741–2758.
[2] A. Araújo, A.K. Das, C. Neves, E. Sousa, Numerical solution for a non-Fickian diffusion in a periodic potential, Commun. Comput. Phys. 13 (2013) 502–525.
[3] H-T. Chen, K-C Liu, Numerical analysis of a non-Fickian diffusion problems in a potential field, Numer Heat Transfer Part B 40 (2001) 265–282.
[4] A. Compte, The generalized Cattaneo equation for the description of anomalous transport processes, J. Phys. A: Math. Gen. 30 (1997) 7277–7289.
[5] M. Cui, High order compact alternating direction implicit method for the generalized sine–Gordon equation, J. Comput. Appl. Math. 235 (2010) 837–849.
[6] A.K. Das, Some non-Fickian diffusion equations: theory and applications, Defect Diffus. Forum 162–163 (1998) 97–118.

[7] D. Deng, C. Zhang, A new fourth-order numerical algorithm for a class of three-dimensional nonlinear evolution equations, Numer. Methods Part D E 29 (2012) 102–130.
[8] D. Deng, C. Zhang, A family of new fourth-order solvers for a nonlinear damped wave equation, Comput. Phys. Commun. 184 (2013) 86–101.
[9] H. Ding, Y. Zhang, A new fourth order compact finite difference scheme for the two dimensional second order hyperbolic equation, J. Comput. Appl. Math. 230 (2009) 626–632.
[10] J. Fort, V. Méndez, Wavefronts in time-delayed reaction-diffusion systems. Theory and comparison to experiment, Rep. Prog. Phys. 65 (2002) 895–954.
[11] H. Goméz, I. Colominas, F. Navarrina, M. Casteleiro, A discontinuous Galerkin method for a hyperbolic model for convection-diffusion problems in CFD, Int. J. Numer. Methods Eng. 71 (2007) 1342–1364.
[12] H. Gómez, I. Colominas, F. Navarrina, M. Casteleiro, A mathematical model and a numerical model for hyperbolic mass transport in compressible flows, Heat Mass Transfer 45 (2008) 219–226.
[13] H. Gómez, I. Colominas, F. Navarrina, J. París, M. Casteleiro, A hyperbolic theory for advection-diffusion problems: mathematical foundations and numerical modeling, Arch. Comput. Methods Eng. 17 (2010) 191–211.
[14] S.M. Hassanizadeh, On the transient non-Fickian dispersion theory, Transp. Porous Media 23 (1996) 107–124.
[15] D.D. Joseph, L. Preziosi, Heat waves, Rev. Mod. Phys. 61 (1989) 41–73.
[16] V. Kulish, K. Poletkin, A generalized relation between the local values of temperature and the corresponding heat flux in a one-dimensional semi-infinite domain with the moving boundary, Int. J. Heat Mass Transfer 55 (2012) 6595–6599.
[17] H. Liao, Z. Sun, Maximum norm error estimates of efficient difference schemes for second-order wave equations, J. Comput. Appl. Math. 235 (2011) 2217–2233.
[18] H.-C. Lin, M.-I. Char, W.-J. Chang, Soret effects on non-Fourier heat and non-Fickian mass diffusion transfer in a slab, Numer. Heat Transfer, Part A 55 (2009) 1096–1115.
[19] K.-C. Liu, H.-T. Chen, Numerical analysis for the hyperbolic heat conduction problem under a pulsed surface disturbance, Appl. Math. Comput. 159 (2004) 887–901.
[20] J. Liu, K. Tang, A new unconditionally stable ADI compact scheme for the two-space-dimensional linear hyperbolic equation, Int. J. Comput. Math. 87 (2010) 2259–2267.
[21] J.E. Macíaz-Díaz, Sufficient conditions for the preservation of the boundedness in a numerical method for a physical model with transport memory and nonlinear damping, Comput. Phys. Commun. 182 (2011) 2471–2478.
[22] J. Mls, L. Herrmann, Oscillations for an equation arising in groundwater flow with the relaxation time, Math. Model. Anal. 16 (2011) 527–536.
[23] R.K. Mohanty, M.K. Jain, An unconditionally stable alternating direction implicit scheme for the two space dimensional linear hyperbolic equation, Numer. Methods Part D E 17 (2001) 684–688.
[24] S. Pankavich, P. Radu, Nonlinear instability of solutions in parabolic and hyperbolic diffusion, Evol. Equ. Control Theory 2 (2013) 403–422.
[25] J. Qin, The new alternating direction implicit difference methods for the wave equation, J. Comput. Appl. Math. 230 (2009) 213–223.
[26] R. Shirmohammadi, A. Moosaie, Non-Fourier heat conduction in a hollow sphere with periodic surface heat flux, Int. Commun. Heat Mass Transfer 36 (2009) 827–833.
[27] A. Sluzalec, Thermal waves propagation in porous material undergoing thermal loading, Int. J. Heat. Mass. Transfer 46 (2003) 1607–1611.
[28] G.A. Sod, Numerical Methods in Fluid Dynamics: Initial and Initial Boundary Value Problems, Cambridge University Press, 1985.
[29] S. Xie, S. Yi, T. Kwon, Fourth-order compact difference and alternating direction implicit schemes for telegraph equations, Comput. Phys. Commun. 183 (2012) 552–569.
[30] M. Yang, Two-time level ADI finite volume method for a class of second-order hyperbolic problems, Appl. Math. Comput. 215 (2010) 3239–3248.
[31] E. Zauderer, Partial Differential Equations of Applied Mathematics, John Wiley & Sons, 1989.
[32] Z. Zhang, An economical difference scheme for heat transport equation at the microscale, Numer. Methods Part D E 20 (2004) 855–863.
[33] Z. Zhang, D. Deng, A new alternating-direction finite element method for hyperbolic equation, Numer. Methods Part D E 23 (2007) 1530–1559.
[34] J. Zhang, J. Zhao, Unconditionally stable finite difference scheme and iterative solution of 2D microscale heat transport equation, J. Comput. Phys. 170 (2001) 261–275.