A fractional diffusion model with resetting

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Abstract. We consider a model that serves as a paradigm for a class of search strategies in which the searcher having explored its environment unsuccessfully for a while, returns to its initial position and begins a new search. The model describes the diffusive motion of a particle, performing a random walk with Lévy distributed jump lengths, which is interrupted at random times when the particle is reset to its initial position. A numerical method is proposed to determine the solutions of this diffusive problem with resetting. The influence of resetting on the solutions is analysed and physical quantities such as the pseudo second moment will be discussed.

1 Introduction

Search problems can occur in many different contexts leading to a variety of interesting processes. Recently, search problems with stochastic resetting of random searchers have been investigated [1–4, 7]. These processes involve two consecutive steps. Firstly, the searcher goes out to search, and secondly, the searcher is drawn back to the starting point.

We consider a one dimensional resetting model, that describes the movement of the searcher in discrete time on a line, starting from an initial position x_0. At time step $t + dt$, the current location $x(t + dt)$ of the searcher is updated via the following stochastic rule: we have $x(t + dt) = x_0$ with probability rdt or $x(t + dt) = x(t) + \eta(t)dt$ with probability $1 - rdt$, where r is the resetting rate. These equations give the probability of a resetting event and the jump lengths described by $\eta(t)$ are independent and identically distributed random variables each drawn from a probability density function that can be a Gaussian white noise with mean zero [4] or a probability density function with a heavy tail [7]. In this work we assume we have a probability density function with a heavy tail. More specifically we consider the class of Lévy stable processes for which the characteristic function is given by $\psi(k) = \mathrm{e}^{-|k|^\alpha}$. The case $\alpha = 2$ corresponds to the Gaussian case described in [4], while the case $0 < \alpha < 2$ describes Lévy flights where the jumps are typically very large.

2 The model

The class of Lévy stable processes described by the characteristic function $\mathrm{e}^{-D|k|^\alpha}$, for $1 < \alpha \leq 2$, can be represented by the fractional diffusion equation [8, 13], that is, the equation for the probability distribution $f(x,t)$, of finding the particle at position x at time t, reads

$$\frac{\partial f(x,t)}{\partial t} = \frac{D}{2\Gamma(2-\alpha)|\cos(\pi\alpha/2)|}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{f(x',t)}{|x'-x|^{\alpha-1}}dx' =: D\frac{\partial^\alpha f}{\partial |x|^\alpha}(x,t), \tag{1}$$

where $\Gamma(\cdot)$ is the Gamma function.

If additionally to the diffusive process we add the resetting events, similarly to what has been presented in [2], we arrive at the equation

$$\frac{\partial f(x,t)}{\partial t} = D\frac{\partial^\alpha f}{\partial |x|^\alpha}(x,t) - rf(x,t) + rf_0(x), \tag{2}$$

where $f_0(x)$ is the initial condition. The second term and third term on the right hand side accounts for the resetting events [2, 9, 12], denoting the negative probability flux $-rf(x,t)$ from each point x and a corresponding positive probability flux into $x = x_0$.

Consider equation (2) and assume $\lim\limits_{x\to\pm\infty} f(x,t) = 0$ and $f_0(x) = \delta(x)$. The resetting rate r should be a positive quantity, on physical grounds. Let f be a sufficiently smooth solution of (2). Then, the characteristic function is given by

$$\hat{f}(k,t) = \mathrm{e}^{-(D|k|^\alpha + r)t} + \frac{r}{D|k|^\alpha + r}(1 - \mathrm{e}^{-(D|k|^\alpha + r)t}). \tag{3}$$

From the inverse Fourier transform, we obtain the following solution of (2),

$$f(x,t) = \mathrm{e}^{-rt}u_\alpha(x,t) + \int_0^t r\mathrm{e}^{-r(t-s)}u_\alpha(x,t-s)ds, \tag{4}$$

where $u_\alpha(x,t)$ is the solution of the fractional diffusion equation (1), which does not include the resetting terms.

The solution of the fractional diffusion equation (1) can be written in closed form, when the initial condition is $f_0(x) = \delta(x)$. It is given in terms of the Fox functions ([8], pag. 27), that is,

$$u_\alpha(x,t) = \frac{1}{\alpha|x|}H^{1,1}_{2,2}\left[\frac{|x|}{(Dt)^{1/\alpha}}\middle|\begin{matrix}(1,1/\alpha) & (1,1/2)\\ (1,1) & (1,1/2)\end{matrix}\right]. \tag{5}$$

It is easy to conclude that for $r > 0$ then $f(x,t) > 0$ for all x,t when u_α is positive and for r positive we have the sum of two positive quantities.

A more general equation can consider a space resetting rate $r(x)$ [2] and can be written as

$$\frac{\partial f}{\partial t}(x,t) = D\frac{\partial^\alpha f}{\partial |x|^\alpha}(x,t) - r(x)f(x,t) + f_0(x)\int_{\mathbb{R}} r(x')f(x',t)dx', \tag{6}$$

where now r depends on x. For r constant we recover (2) when $\int_{\mathbb{R}} f(x',t)dx' = 1$.

In the next section we derive a numerical method to solve the fractional diffusion equation with resetting terms (6) and analyze its efficiency. Then, in the last section, we discuss the influence of resetting in the moments and how it changes the character of the solution.

3 The numerical method

The equation under consideration can be written in the form

$$\frac{\partial f}{\partial t}(x,t) = \frac{D}{2|\cos(\alpha\pi/2)|}\left(\frac{\partial^\alpha f}{\partial x^\alpha} + \frac{\partial^\alpha f}{\partial(-x)^\alpha}\right)(x,t) - r(x)f(x,t) + s(x,t), \quad (7)$$

where $s(x,t) = f_0(x)\int_{\mathbb{R}} r(x')f(x',t)dx'$ and the fractional Riesz operator has been represented in terms of the left and right Riemann-Liouville derivatives of order α, $1 < \alpha < 2$. The left and right Riemann-Liouville fractional derivatives of order α, for $x \in [a,b]$, $-\infty \leq a < b \leq \infty$ are given respectively by

$$\frac{\partial^\alpha f}{\partial x^\alpha}(x,t) = \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_a^x f(\xi,t)(x-\xi)^{1-\alpha}d\xi, \quad (8)$$

$$\frac{\partial^\alpha f}{\partial(-x)^\alpha}(x,t) = \frac{(-1)^2}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_x^b f(\xi,t)(\xi-x)^{1-\alpha}d\xi. \quad (9)$$

Our problem is defined in the domain $\mathbb{R} \times [0,T]$. We define a uniform discrete domain in space and time, that is, $x_{j+1} = x_j + \Delta x,\ j \in \mathbb{Z},\quad t_n = n\Delta t,\quad \Delta t = T/M$, where Δx is the space step and Δt is the time step.

Let f_j^n denote the approximation of $f(x_j,t_n)$. We approximate the fractional Riesz operator using the approximations discussed in [10, 11] which are approximations of the left and right Riemann-Liouville derivatives. These approximations are respectively given by the discrete operators $\delta_l^\alpha f(x_j,t)/\Delta x^\alpha$ and $\delta_r^\alpha f(x_j,t)/\Delta x^\alpha$ where

$$\delta_l^\alpha f(x_j,t) = \sum_{m=-1}^{\infty}\frac{q_m}{\Gamma(4-\alpha)}f(x_{j-m},t), \quad \delta_r^\alpha f(x_j,t) = \sum_{m=-1}^{\infty}\frac{q_m}{\Gamma(4-\alpha)}f(x_{j+m},t). \quad (10)$$

The coefficients q_m are defined by

$$q_{-1} = a_0, \quad q_0 = -2a_0 + a_1 \quad q_m = a_{m-1} - 2a_m + a_{m+1},\ m \geq 1, \quad (11)$$

where $a_0 = 1$ and $a_m = (m+1)^{3-\alpha} - 2m^{3-\alpha} + (m-1)^{3-\alpha},\ m \geq 1$. Finally, the discrete operator $\delta^\alpha f/\Delta x^\alpha$ approximates the Riesz operator, where

$$\delta^\alpha f(x_j,t) = \frac{1}{2|\cos(\alpha\pi/2)|}\left(\delta_l^\alpha f(x_j,t) + \delta_r^\alpha f(x_j,t)\right). \quad (12)$$

We consider the explicit first order upwind numerical method

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = D\frac{\delta^\alpha f_j^n}{(\Delta x)^\alpha} - r_j^n f_j^n + s_j^n, \quad (13)$$

where $r_j^n = r(x_j,t_n)$ and s_j^n is the second order approximation of $s(x_j,t_n)$

$$s_j^n = f_0(x_j)\Delta x\left[\frac{1}{2}(r_0 f_0^n + r_N f_N^n) + \sum_{j=1}^{N-1} r_j f_j^n\right]. \quad (14)$$

We discuss the efficiency of the numerical method in terms of two essential aspects, which are accuracy and stability.

The accuracy of the method comes directly from its formulation and the fact that the approximation we have used for the Riesz operator is known to be second order accurate. This conclusion follows from the next result concerning the approximation of the left fractional derivative.

Theorem 1 *[10, 11]: Let u be a function with sufficiently many continuous spatial derivatives that vanish at infinity in an appropriate manner. Then, we have that*

$$\frac{\partial^\alpha u}{\partial x^\alpha}(x_j) - \frac{\delta_l^\alpha u}{(\Delta x)^\alpha}(x_j) = O((\Delta x)^2).$$

A similar result is valid for the right fractional derivative. From here we can conclude that the Riesz operator is approximated by a second order accurate formula.

In order to derive stability conditions for the finite difference scheme, we apply the von Neumann analysis or Fourier analysis. We have assumed resetting r locally constant. Let

$$s_\alpha = -\sum_{m=-1}^{\infty} q_m \cos(m\theta), \qquad A_\alpha = \frac{\mu_\alpha}{2|\cos(\alpha\pi/2)|}, \qquad \mu_\alpha = \frac{D\Delta t}{(\Delta x)^\alpha}.$$

Theorem 2 *The numerical method (13) is von Neumann stable if and only if*

$$s_\alpha A_\alpha \leq 2. \tag{15}$$

Proof. Fourier analysis assumes that for a solution defined in the whole real line, the error will be propagated forward in time according to the equation

$$e_j^{n+1} = e_j^n + \mu_\alpha \delta^\alpha e_j^n - r e_j^n. \tag{16}$$

This analysis also assumes the error e_j^n is decomposed into a Fourier series with terms given by $\kappa_p^n e^{i\xi_p(j\Delta x)}$, where κ_p^n is the amplitude of the p-th harmonic and θ is the phase angle and covers the domain $[-\pi, \pi]$. Considering a single mode $\kappa^n e^{ij\theta}$, its time evolution is determined by the same numerical scheme as the error e_j^n. The stability conditions will be satisfied if the amplification factor κ does not grow in time.

We denote by $\kappa(\theta; \mu_\alpha)$ the amplification factor since it will depend on μ_α. If we insert $\kappa^n e^{ij\theta}$ in (16) we obtain the equality for the amplification factor

$$\kappa(\theta;\mu_\alpha) = 1 - r\Delta t + \frac{\mu_\alpha}{2|\cos(\alpha\pi/2)|\Gamma(4-\alpha)}\left[\sum_{m=-1}^{\infty} q_m \mathrm{e}^{-im\theta} + \sum_{m=-1}^{\infty} q_m \mathrm{e}^{im\theta}\right]$$
$$= 1 - r\Delta t + \frac{\mu_\alpha}{2|\cos(\alpha\pi/2)|\Gamma(4-\alpha)}\left[\sum_{m=-1}^{\infty} q_m \cos(m\theta)\right] = 1 - r\Delta t - A_\alpha s_\alpha.$$

Hence, $|\kappa(\theta;\mu_\alpha)| = |1 - A_\alpha s_\alpha - r\Delta t| \leq |1 - A_\alpha s_\alpha| + r\Delta t$. Note that $s_\alpha \geq 0$ (see [11]). Therefore the method is stable if and only if $|1 - A_\alpha s_\alpha| \leq 1$. This inequality is equivalent to $s_\alpha A_\alpha \leq 2$. □

Note that $s_\alpha \leq |q_{-1} + q_1| + |q_0| + |\sum_{m=2}^{\infty} q_m| \leq 8(1 - 2^{1-\alpha}) \leq 8$ and if $A_\alpha \leq 1/4$ the method is stable for all $1 < \alpha \leq 2$. Therefore the numerical method can be implemented efficiently by considering a space step Δx and a time step Δt that verifies $A_\alpha = 1/4$.

The tests done in the next section for the solution defined in all real line are run in a domain $[-L, L]$. The constant L should be large enough such that the presence of the artificial boundaries at $x = \pm L$ do not affect the accuracy of the approximate solution. In particular, the power law decay verified for anomalous diffusion requires some additional care. Without resetting this would necessarily mean to increase the computational domain as we increase time.

The matricial form of the numerical method is build in the domain $[-L, L]$ and the discrete points in space are $x_j = -L + j\Delta x,\ \Delta x = 2L/N$. We get

$$\mathbf{f}^{n+1} = (\mathbf{I} + \mathbf{A}_\alpha - \mathbf{R} * \mathbf{I})\mathbf{f}^n + \mathbf{s}^n, \tag{17}$$

where $\mathbf{A}_\alpha$ is an $(N-1)\times(N-1)$ matrix related to the anomalous diffusion term, $\mathbf{R}$ is a diagonal matrix with the resetting values r_j as entries and $\mathbf{s}^n$ is the vector that contains the entries s_j^n. The matrix $\mathbf{A}_\alpha$ will be of the form $\mathbf{A}_\alpha = \mathbf{L}_\alpha + \mathbf{L}_\alpha^T$, where $\mathbf{L}_\alpha$ is a matrix that contains the coefficients defined in (11).

4 The effect of resetting

In this section we discuss the effect of resetting in the solutions. We consider the initial condition $\delta_\epsilon(x) = (1/\epsilon\sqrt{\pi})\mathrm{e}^{-(x-x_0)^2/\epsilon^2}$ that can be seen as an approximation of the delta Dirac function.

In the absence of resetting, the anomalous or classical diffusion in free space does not have a stationary state (see Figure 1(a) and Figure 1(b)). However a nonzero rate of resetting to a fixed position leads to a stationary state when $t \to \infty$ (see Figure 1(c) and Figure 1(d)). The steady state solution of (7), when $r(x)$ is constant is $f_{st}(x) = \sqrt{r/D}\exp(-\sqrt{r/D}|x - x_0|)$. The numerical solution, when $\alpha = 2$, plotted in Figure 1(d) is in agreement to this steady state solution. This stationary distribution is non-gaussian in x, contrasting with the fact that if we omit the resetting term, then the stationary solution would be gaussian in x. Therefore, the resetting term changes the character of the long time solution.

For the sake of clarity in the discussion that follows we assume $x_0 = 0$. It is well known that the first and second moments for the classical diffusion are given respectively by $< x >= 0$ and $< x^2 >= 2Dt$. If we consider the diffusion problem with resetting, for $\alpha = 2$, we can find, from the characteristic function given by (3), that the first and second moments are given respectively by

$$M_1(t) := \int_{\mathbb{R}} x f(x,t)dx = 0, \quad M_2(t) := \int_{\mathbb{R}} x^2 f(x,t)dx = \frac{2D}{r}\left(1 - \mathrm{e}^{-rt}\right). \tag{18}$$

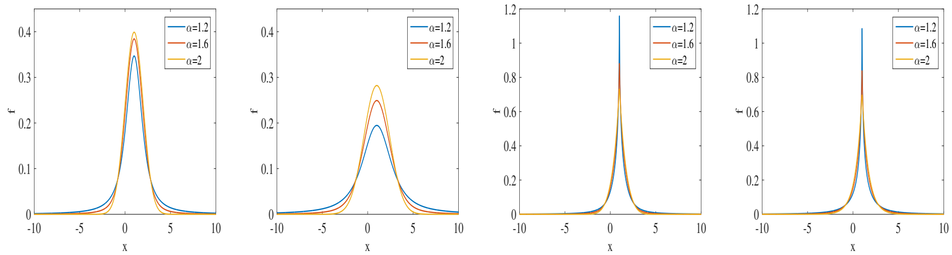

Fig. 1. Plots of $f(x,t)$ for $x_0 = 1$, $D = 0.5$ computed with $\Delta x = 0.005$, $\Delta t = 0.01\Delta x$. Left figures: $r(x) = 0$. (a) $t = 1$; (b) $t = 2$. Right figures: $r(x) = 1$; (c) $t = 1$; (d) $t = 5$. For $r(x) = 1$ and $t = 10$ the solution is similar to (d) indicating we have reached the steady state.

At this point we can ask two questions, for the case $\alpha = 2$. Can we see at which time the steady state is reached by looking at the second moment? And if we increase the resetting rate does it mean the solution is pushed quicker to the steady state? In order to answer to the previous questions in Figure 2 we

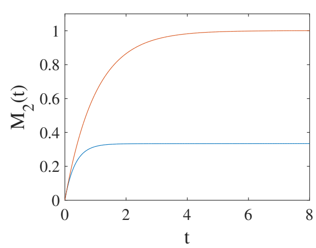

Fig. 2. Second moment $M_2(t)$ for the classical diffusion with resetting $r = 1$ (red) and $r = 3$ (blue): $D = 0.5$.

plot the second moment $M_2(t)$ for the classical diffusion with resetting $r = 1$ and $r = 3$. Theoretically, we have that

$$\lim_{t\to\infty} \frac{2D}{r}(1 - \mathrm{e}^{-rt}) = \frac{2D}{r} \tag{19}$$

and this is in agreement to what is shown in Figure 2. We can observe that for $r = 1$ the second moment starts to look constant around $t = 6$ and according to the value $M_2(t) = 2D/r = 1$. For $r = 3$ is around $t = 2$ where $M_2(t) = 1/3$ which is also according to (19).

Now let us turn to the anomalous diffusion case. For $1 < \alpha < 2$ the second moment for diffusion without resetting is known to be divergent and the first moment is zero. For the case with resetting from the characteristic function, given by (3), we can also conclude that the second moment is divergent and the first moment is zero. The problem of having a diverging second moment encountered in the discussions of Lévy flights can be circumvented by considering a pseudo second moment, that is,

$$M_2^L(t) := < x^2 >_L = \int_{L_1(t)}^{L_2(t)} x^2 f(x,t) dx \tag{20}$$

according to which the walker is considered in an imaginary box with $L = \max\{|L_1|, |L_2|\}$. Without loss of generality, we assume a symmetric box, that is, $L_2(t) = -L_1(t) = L(t)$. Note that the cut-offs of the integral are time dependent and the imaginary box is chosen in the spatial interval. We can say it gives a measure, that a finite portion of the probability is gathered within the given interval $2L(t)$. Pseudo second moments have been considered before in literature. For instance, in [5, 6, 8], the box considered was $[L_1 t^{1/\alpha}, L_2 t^{1/\alpha}]$. Here, we consider a different type, that is, for a final time T we consider a sufficently large box $[-L(T), L(T)]$ and then we compute (20) in that box for all $0 < t \leq T$.

Since the density follows the power-law asymptotic behaviour the cut-offs of the integral needs to be chosen such that the asymptotic behaviour of $f(x,t)$ is reached. This can be done, by choosing a box $[-L, L]$ for which the values of $f(x,t)$ are very small in $\pm L$. Note that for smaller values of α we have larger tails, that is, the tail increases with decreasing α.

In what follows we compare the estimates of the pseudo second moments, $M_2^L(t)$, for the problems with and without resetting. We plot in Figure 3(a) and Figure 3(b) the results for $r = 0$ and in Figure 3(c) and 3(d) we plot the results for $r = 1$. From these results we infer that for the problem without resetting we have

$$M_2^L(t) = C_\alpha t \tag{21}$$

with C_α a constant that depends on α. Then in the presence of resetting we get the behaviour

$$M_2^L(t) \sim C_\alpha \left(\frac{1 - \exp(-rt)}{r} \right). \tag{22}$$

Another interesting information we can obtain from the pseudo second moments is related to the steady state solutions. The results suggest that in the presence of resetting we reach a steady state for all α's as can be seen in Figure 3. It also seems that smaller is the value of α, it takes longer to reach the steady state. This information was obtained for particular values of L. Although the pseudo moment increases with L as expected, it becomes constant at similar instants of time for different values of L.

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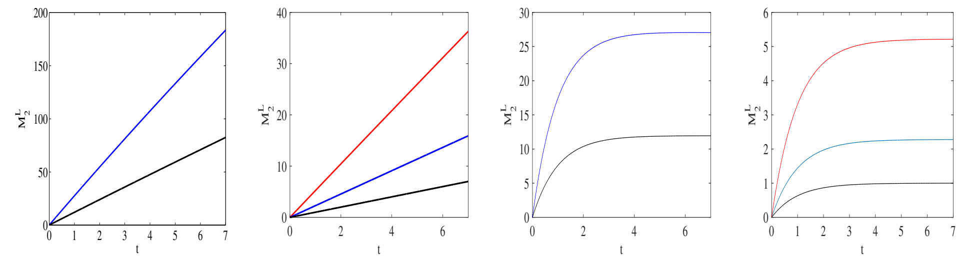

Fig. 3. Left figures: Pseudo second moment for anomalous diffusion with resetting $r = 0$ with $L = 100$. (a) $\alpha = 1.2$ (blue), $\alpha = 1.4$ (black). (b) $\alpha = 1.6$ (red), $\alpha = 1.8$ (blue), $\alpha = 2$ (black). Right figures: Pseudo second moment for anomalous diffusion with resetting $r = 1$ with $L = 100$: (c) $\alpha = 1.2$ (blue), $\alpha = 1.4$ (black); (d) $\alpha = 1.6$ (red), $\alpha = 1.8$ (blue), $\alpha = 2$ (black).

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