On (co)normal closure operators

Gonçalo Gutierres – CMUC/Universidade de Coimbra

Joint work with Maria Manuel Clementino

Given a class of groups $\mathbb{A} \subseteq \mathsf{Grp}$, the normal closure induced by $\mathbb{A}$ is given by

$$\mathrm{norm}_G^{\mathbb{A}}(H) := \bigcap \{N \mid H \subseteq N \lhd G,\ G/N \subseteq A \in \mathbb{A}\} = \bigcap \{f^{-1}(0) \mid f : G \to A \in \mathbb{A},\ f(H) = 0\}.$$

It is easy to see that the normal closure can be defined in any category with a 0-object and an $\mathcal{M}$-right factorization, where $\mathcal{M}$ contains all normal monomorphisms.

It is patent that the constructions of the normal and the regular closure are very similar. Accordingly, we will define the conormal closure - in parallel with the coregular closure - and, using a unifying setting, we will generalize results obtained for regular/coregular closures in [CT].

$\mathbb{X}$ - category with finite limits and a factorization system $(\mathcal{E}, \mathcal{M})$ with $\mathcal{M} \subseteq$ *Mono*$\mathbb{X}$.

$\mathbb{X}$ - category with finite limits and a factorization system $(\mathcal{E}, \mathcal{M})$ with $\mathcal{M} \subseteq$ *Mono*$\mathbb{X}$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e"'] & & Y \\ & f(X) \arrow[ur, "m"'] & \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "m"] \arrow[dr] & X \arrow[r, "f"] & Y \\ & f(M) \arrow[ur, "f(m)"'] & \end{tikzcd}$$

$\mathbb{X}$ - category with finite limits and a factorization system $(\mathcal{E}, \mathcal{M})$ with $\mathcal{M} \subseteq$ *Mono*$\mathbb{X}$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e"'] & & Y \\ & f(X) \arrow[ur, "m"'] & \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "m"] \arrow[dr] & X \arrow[r, "f"] & Y \\ & f(M) \arrow[ur, "f(m)"'] & \end{tikzcd}$$

$$\begin{tikzcd} f^{-1}(M) \arrow[r] \arrow[d, "f^{-1}(m)"'] & M \arrow[d, "m"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

A closure operator c on $\mathbb{X}$ w.r.t. $\mathcal{M}$ is a family of functions $(c_X : \mathcal{M}/X \to \mathcal{M}/X)_{X\in\mathbb{X}}$:

1. $m \leq c_X(m)$;
2. if $m \leq m'$ then $c_X(m) \leq c_X(m')$;
3. $f(c_X(m)) \leq c_Y(f(m))$ for $f : X \to Y$, $m \in \mathcal{M}/X$.

A closure operator c on $\mathbb{X}$ w.r.t. $\mathcal{M}$ is a family of functions $(c_X : \mathcal{M}/X \to \mathcal{M}/X)_{X \in \mathbb{X}}$:

1. $m \leq c_X(m)$;
2. if $m \leq m'$ then $c_X(m) \leq c_X(m')$;
3. $f(c_X(m)) \leq c_Y(f(m))$ for $f : X \to Y$, $m \in \mathcal{M}/X$.

$c_X(f^{-1}(m)) \leq f^{-1}(c_Y(m))$ for $f : X \to Y$, $m \in \mathcal{M}/Y$

A closure operator c on $\mathbb{X}$ w.r.t. $\mathcal{M}$ is a family of functions $(c_X : \mathcal{M}/X \to \mathcal{M}/X)_{X \in \mathbb{X}}$:

1. $m \leq c_X(m)$;
2. if $m \leq m'$ then $c_X(m) \leq c_X(m')$;
3. $f(c_X(m)) \leq c_Y(f(m))$ for $f : X \to Y$, $m \in \mathcal{M}/X$.

$c_X(f^{-1}(m)) \leq f^{-1}(c_Y(m))$ for $f : X \to Y$, $m \in \mathcal{M}/Y$

c is idempotent if $c(c(m)) = c(m)$ for all $m \in \mathcal{M}$.

c is weakly hereditary if $c(j_m) = 1_{cM}$ for all $m \in \mathcal{M}$ with $c(m) \cdot j_m = m$.

$\mathbb{X}$ - category with a 0 object and $\mathcal{M} \supseteq$ *NormalMonos*

$\mathbb{X}$ - category with a 0 object and $\mathcal{M} \supseteq$ *NormalMonos*

$$\mathrm{norm}_X^{\mathbb{A}}(m) = \bigwedge \{ f^{-1}(0) \mid f : X \to A \in \mathbb{A},\ m \leq f^{-1}(0_A) \}$$

$\mathbb{X}$ - category with a 0 object and $\mathcal{M} \supseteq$ *NormalMonos*

$$\mathrm{norm}_X^{\mathbb{A}}(m) = \bigwedge \{ f^{-1}(0) \mid f : X \to A \in \mathbb{A},\ m \leq f^{-1}(0_A) \}$$

$$\mathcal{F}(c) = \{ X \mid c_X(0) = 0 \}$$

$\mathbb{X}$ - category with a 0 object and $\mathcal{M} \supseteq$ *NormalMonos*

$$\mathrm{norm}_X^{\mathbb{A}}(m) = \bigwedge \{ f^{-1}(0) \mid f : X \to A \in \mathbb{A},\ m \leq f^{-1}(0_A) \}$$

$$\mathcal{F}(c) = \{ X \mid c_X(0) = 0 \}$$

$$\begin{tikzcd} \mathrm{S}(\mathbb{X})^{op} \arrow[r, shift left, "\mathrm{norm}"] & \mathrm{CL}(\mathbb{X}, \mathcal{M}) \arrow[l, shift left, "\mathcal{F}"] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge] \mathrm{CL}(\mathbb{X}, \mathcal{M}) \arrow[r, shift left, "\mathcal{T}"] & \mathrm{S}(\mathbb{X}) \arrow[l, shift left, "\mathrm{conorm}"] \end{tikzcd}$$

$$
\begin{tikzcd}
\mathrm{CL}(\mathbb{X},\mathcal{M}) \arrow[r, shift left, "\mathcal{T}"] & \mathrm{S}(\mathbb{X}) \arrow[l, shift left, "\mathrm{conorm}"]
\end{tikzcd}
$$

$$\mathcal{T}(c) = \{X \mid c_X(0) = 1_X\}$$

$$\mathrm{conorm}^{\mathbb{A}}_X(m) = m \vee \bigvee \{f(1_A) | f : A \to X,\ A \in \mathbb{A},\ f(0) \leq m\}$$

$$
\begin{tikzcd}
 & \mathrm{CL}(\mathbb{X}, \mathcal{M}) \arrow[dl, shift left, "\mathcal{F}"] \arrow[dr, shift left, "\mathcal{T}"] & \\
\mathrm{S}(\mathbb{X})^{op} \arrow[ur, shift left, "\mathrm{norm}"] \arrow[rr, shift right, "L"'] & & \mathrm{S}(\mathbb{X}) \arrow[ul, shift left, "\mathrm{conorm}"] \arrow[ll, shift right, "R"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathrm{CL}(\mathbb{X}, \mathcal{M}) \arrow[dl, shift left, "\mathcal{F}"] \arrow[dr, shift left, "\mathcal{T}"] & \\
\mathrm{S}(\mathbb{X})^{op} \arrow[ur, shift left, "\mathrm{norm}"] \arrow[rr, shift right, "L"'] & & \mathrm{S}(\mathbb{X}) \arrow[ll, shift right, "R"'] \arrow[ul, shift left, "\mathrm{conorm}"]
\end{tikzcd}
$$

$$R(\mathbb{A}) = \{X \mid (\forall f : X \to A,\ A \in \mathbb{A})\, f(X) = 0\}$$

$$L(\mathbb{A}) = \{X \mid (\forall f : A \to X,\ A \in \mathbb{A})\, f(A) = 0\}$$

$\mathcal{M} \supseteq \mathit{RegularMonos}$

$\mathcal{M} \supseteq$ *RegularMonos*

$$\mathrm{reg}_X^{\mathbb{A}}(m) = \bigwedge \{ f^{-1}(\delta_A) \mid f : X \to A^2,\ A \in \mathbb{A}, m \leq f^{-1}(\delta_A) \}$$

$$\mathrm{coreg}_X^{\mathbb{A}}(m) = m \vee \bigvee \{ f(1_{A^2}) | f : A^2 \to X,\ A \in \mathbb{A},\ f(\delta_A) \leq m \}$$

$\mathcal{M} \supseteq$ *RegularMonos*

$$\mathrm{reg}^{\mathbb{A}}_X(m) = \bigwedge \{ f^{-1}(\delta_A) \mid f : X \to A^2,\ A \in \mathbb{A}, m \leq f^{-1}(\delta_A) \}$$

$$\mathrm{coreg}^{\mathbb{A}}_X(m) = m \vee \bigvee \{ f(1_{A^2}) | f : A^2 \to X,\ A \in \mathbb{A},\ f(\delta_A) \leq m \}$$

$$\Delta(c) = \{ X \mid c(\delta_X) = \delta_X \} \qquad \nabla(c) = \{ X \mid c(\delta_X) = 1_{X^2} \}$$

$\mathcal{M} \supseteq$ *RegularMonos*

$$\mathrm{reg}^{\mathbb{A}}_X(m) = \bigwedge\{f^{-1}(\delta_A) \mid f : X \to A^2,\ A \in \mathbb{A}, m \leq f^{-1}(\delta_A)\}$$

$$\mathrm{coreg}^{\mathbb{A}}_X(m) = m \vee \bigvee\{f(1_{A^2}) | f : A^2 \to X,\ A \in \mathbb{A},\ f(\delta_A) \leq m\}$$

$$\Delta(c) = \{X \mid c(\delta_X) = \delta_X\} \qquad \nabla(c) = \{X \mid c(\delta_X) = 1_{X^2}\}$$

$$\mathrm{S}(\mathbb{X})^{op} \underset{\Delta}{\overset{\mathrm{reg}}{\rightleftarrows}} \mathrm{CL}(\mathbb{X}, \mathcal{M}) \underset{\mathrm{coreg}}{\overset{\nabla}{\rightleftarrows}} \mathrm{S}(\mathbb{X})$$

$$r(\mathbb{A}) = \{X \mid (\forall f : X \to A,\ A \in \mathbb{A})\ f(X) \text{ is preterminal}\}$$

$$l(\mathbb{A}) = \{X \mid (\forall f : A \to X,\ A \in \mathbb{A})\ f(A) \text{ is preterminal}\}$$

$r(\mathbb{A}) = \{X \mid (\forall f : X \to A,\ A \in \mathbb{A})\ f(X)$ is preterminal$\}$

$l(\mathbb{A}) = \{X \mid (\forall f : A \to X,\ A \in \mathbb{A})\ f(A)$ is preterminal$\}$

A point of X is $(x : 1 \to X) \in \mathcal{M}$ with 1 the terminal object.

$$r(\mathbb{A}) = \{X \mid (\forall f : X \to A,\ A \in \mathbb{A})\ f(X) \text{ is preterminal}\}$$

$$l(\mathbb{A}) = \{X \mid (\forall f : A \to X,\ A \in \mathbb{A})\ f(A) \text{ is preterminal}\}$$

A point of X is $(x : 1 \to X) \in \mathcal{M}$ with 1 the terminal object.

$1_X = \bigvee\{x \mid x \text{ is a point of } X\}$ for all $X \in \mathbb{X}$
$\mathcal{E}$ is closed under the formation of squares.

$$\begin{tikzcd}
 & \mathrm{CL}(\mathbb{X}, \mathcal{M}) \arrow[dl, shift left, "\Delta"] \arrow[dr, shift left, "\nabla"] & \\
\mathrm{S}(\mathbb{X})^{op} \arrow[ur, shift left, "\mathrm{reg}"] \arrow[rr, shift right, "l"'] & & \mathrm{S}(\mathbb{X}) \arrow[ul, shift left, "\mathrm{coreg}"] \arrow[ll, shift right, "r"']
\end{tikzcd}$$

$\mathcal{N} \subseteq \mathcal{M}$

$$
\begin{tikzcd}
 & \mathrm{CL}(\mathbb{X}, \mathcal{M}) \arrow[dl, shift left, "D_{\mathcal{N}}"] \arrow[dr, shift left, "I_{\mathcal{N}}"] & \\
\mathrm{S}(\mathbb{X})^{op} \arrow[ur, shift left, "T_{\mathcal{N}}"] \arrow[rr, shift right, "L_{\mathcal{N}}"'] & & \mathrm{S}(\mathbb{X}) \arrow[ul, shift left, "J_{\mathcal{N}}"] \arrow[ll, shift right, "R_{\mathcal{N}}"']
\end{tikzcd}
$$

$$\mathcal{N} \subseteq \mathcal{M}$$

$$\begin{tikzcd}
& \mathrm{CL}(\mathbb{X},\mathcal{M}) \arrow[dl, shift left, "D_{\mathcal{N}}"] \arrow[dr, shift left, "I_{\mathcal{N}}"] & \\
\mathrm{S}(\mathbb{X})^{op} \arrow[ur, shift left, "T_{\mathcal{N}}"] \arrow[rr, shift right, "L_{\mathcal{N}}"'] & & \mathrm{S}(\mathbb{X}) \arrow[ul, shift left, "J_{\mathcal{N}}"] \arrow[ll, shift right, "R_{\mathcal{N}}"']
\end{tikzcd}$$

$$D_{\mathcal{N}}(c) = \{X \mid \forall n \in \mathcal{N}/X \;\; c(n) = n\}$$

$$T_{\mathcal{N}}(\mathbb{A})_X(m) = \bigwedge\{f^{-1}(n) | f : X \to A \in \mathbb{A},\ n \in \mathcal{N}/A,\ m \leq f^{-1}(n)\}$$

$$I_{\mathcal{N}}(c) = \{X \mid \forall n \in \mathcal{N}/X \;\; c(n) = 1_X\}$$

$$J_{\mathcal{N}}(\mathbb{A})_X(m) = m \vee \bigvee\{f(1_A) | f : A \to X,\ A \in \mathbb{A},\ (\exists n \in \mathcal{N})\ f(n) \leq m\}$$

Examples

$$\mathcal{Z} = \{0_X \mid 0_X : 0 \to X\}$$

$T_{\mathcal{Z}} = \mathrm{norm}$ and $J_{\mathcal{Z}} = \mathrm{conorm}$.

$D_{\mathcal{Z}} = \mathcal{F}$ and $I_{\mathcal{Z}} = \mathcal{T}$.

Examples

$$\mathcal{Z} = \{0_X \mid 0_X : 0 \longrightarrow X\}$$

$T_{\mathcal{Z}} = \mathrm{norm}$ and $J_{\mathcal{Z}} = \mathrm{conorm}$.

$D_{\mathcal{Z}} = \mathcal{F}$ and $I_{\mathcal{Z}} = \mathcal{T}$.

$$\mathcal{D} = \{\delta_X \mid \delta_X : X \longrightarrow X^2\}$$

$\mathbb{A} \subseteq \mathbb{X}$ $\qquad$ $\mathbb{A}^2 = \{A^2 \mid A \in \mathbb{A}\}$

$T_{\mathcal{D}}(\mathbb{A}^2) = \mathrm{reg}^{\mathbb{A}}$ and $J_{\mathcal{D}}(\mathbb{A}^2) = \mathrm{coreg}^{\mathbb{A}}$.

$A \in \Delta(c) \Leftrightarrow A^2 \in D_{\mathcal{D}}$

$A \in \nabla(c) \Leftrightarrow A^2 \in I_{\mathcal{D}}$

$$
\begin{tikzcd}
\mathrm{S}(\mathcal{M})^{op} \arrow[r, shift left, "T"] & \mathrm{CL}(\mathbb{X}, \mathcal{M}) \arrow[l, shift left, "D"] \arrow[r, shift left, "I"] & \mathrm{S}(\mathcal{M}) \arrow[l, shift left, "J"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{S}(\mathcal{M})^{op} \arrow[r, shift left, "T"] & \mathrm{CL}(\mathbb{X},\mathcal{M}) \arrow[l, shift left, "D"] \arrow[r, shift left, "I"] & \mathrm{S}(\mathcal{M}) \arrow[l, shift left, "J"]
\end{tikzcd}
$$

$$D(c) = \{m \mid c(m) = m\}$$

$$T(\mathcal{A})_X(m) = \bigwedge\{f^{-1}(n) | f : X \to A \in \mathbb{A},\ n \in \mathcal{A},\ m \leq f^{-1}(n)\}$$

$$I(c) = \{X \mid c(m) = 1_X\}$$

$$J(\mathcal{A})_X(m) = m \vee \bigvee\{f(1_A) | f : A \to X,\ A \in \mathbb{A},\ (\exists n \in \mathcal{A})\, f(n) \leq m\}$$

$$\begin{tikzcd}[column sep=huge] \mathrm{S}(\mathcal{M})^{op} \arrow[r, shift left, "T"] & \mathrm{CL}(\mathbb{X}, \mathcal{M}) \arrow[l, shift left, "D"] \arrow[r, shift left, "I"] & \mathrm{S}(\mathcal{M}) \arrow[l, shift left, "J"] \end{tikzcd}$$

$$\mathrm{S}(\mathcal{M})^{op} \underset{D}{\overset{T}{\rightleftarrows}} \mathrm{CL}(\mathbb{X}, \mathcal{M}) \underset{J}{\overset{I}{\rightleftarrows}} \mathrm{S}(\mathcal{M})$$

$c = JI(c)$ if and only if c is weakly hereditary.

$$
\begin{tikzcd}
\mathrm{S}(\mathcal{M})^{op} \arrow[r, shift left, "T"] & \mathrm{CL}(\mathbb{X}, \mathcal{M}) \arrow[l, shift left, "D"] \arrow[r, shift left, "I"] & \mathrm{S}(\mathcal{M}) \arrow[l, shift left, "J"]
\end{tikzcd}
$$

$c = JI(c)$ if and only if c is weakly hereditary.

$c = TD(c)$ if and only if c is idempotent.

$$\begin{tikzcd} \mathrm{S}(\mathcal{M})^{op} \arrow[r, shift left, "T"] & \mathrm{CL}(\mathbb{X},\mathcal{M}) \arrow[l, shift left, "D"] \arrow[r, shift left, "I"] & \mathrm{S}(\mathcal{M}) \arrow[l, shift left, "J"] \end{tikzcd}$$

$c = JI(c)$ if and only if c is weakly hereditary.

$c = TD(c)$ if and only if c is idempotent.

$\mathcal{A} = IJ(\mathcal{A})$ iff $\mathbb{A}$ is an $\mathcal{M}$-coreflective subcategory of $\mathcal{M}/\mathbb{X}$.

$$\begin{tikzcd} \mathrm{S}(\mathcal{M})^{op} \arrow[r, shift left, "T"] & \mathrm{CL}(\mathbb{X}, \mathcal{M}) \arrow[l, shift left, "D"] \arrow[r, shift left, "I"] & \mathrm{S}(\mathcal{M}) \arrow[l, shift left, "J"] \end{tikzcd}$$

$c = JI(c)$ if and only if c is weakly hereditary.

$c = TD(c)$ if and only if c is idempotent.

$\mathcal{A} = IJ(\mathcal{A})$ iff $\mathbb{A}$ is an $\mathcal{M}$-coreflective subcategory of $\mathcal{M}/\mathbb{X}$.

$\mathcal{A} = DT(\mathcal{A})$ iff and only if $\mathbb{A}$ is an *Iso*$\mathbb{X}$-reflective subcategory of $\mathcal{M}/\mathbb{X}$.

A Torsion Theory?

Two morphisms $m, n \in \mathcal{M}$ are orthogonal $m \perp n$ if $f(m) \leq n \Rightarrow f(1_X) \leq n$ for $f : X \to Y$, $m \in \mathcal{M}/X$, $n \in \mathcal{M}/Y$.

A Torsion Theory?

Two morphisms $m, n \in \mathcal{M}$ are orthogonal $m \perp n$ if $f(m) \leq n \Rightarrow f(1_X) \leq n$ for $f : X \to Y$, $m \in \mathcal{M}/X$, $n \in \mathcal{M}/Y$.

A 'torsion theory' in $\mathcal{M}$ is a pair $(\mathcal{A}, \mathcal{B})$ such that:

1. for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, $a \perp b$;
2. for every $m \in \mathcal{M}$, there is $a \in \mathcal{A}$, $b \in \mathcal{B}$ such that $m = b \cdot a$.

A Torsion Theory?

Two morphisms $m, n \in \mathcal{M}$ are orthogonal $m \perp n$ if
$f(m) \leq n \Rightarrow f(1_X) \leq n$ for $f : X \to Y$, $m \in \mathcal{M}/X$, $n \in \mathcal{M}/Y$.

A 'torsion theory' in $\mathcal{M}$ is a pair $(\mathcal{A}, \mathcal{B})$ such that:

1. for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, $a \perp b$;
2. for every $m \in \mathcal{M}$, there is $a \in \mathcal{A}$, $b \in \mathcal{B}$ such that $m = b \cdot a$.

2 is equivalent to saying that $J(\mathcal{A}) = T(\mathcal{B})$.

A Torsion Theory?

Two morphisms $m, n \in \mathcal{M}$ are orthogonal $m \perp n$ if $f(m) \leq n \Rightarrow f(1_X) \leq n$ for $f : X \to Y$, $m \in \mathcal{M}/X$, $n \in \mathcal{M}/Y$.

A 'torsion theory' in $\mathcal{M}$ is a pair $(\mathcal{A}, \mathcal{B})$ such that:

1. for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, $a \perp b$;
2. for every $m \in \mathcal{M}$, there is $a \in \mathcal{A}$, $b \in \mathcal{B}$ such that $m = b \cdot a$.

2 is equivalent to saying that $J(\mathcal{A}) = T(\mathcal{B})$.

The pair $(\mathcal{A}, \mathcal{B})$ is a torsion theory if and only if there is an idempotent weakly hereditary closure operator c such that $\mathcal{A} = I(c)$ and $\mathcal{B} = D(c)$.

$$
\begin{tikzcd}
 & \mathrm{CL}(\mathbb{X}, \mathcal{M}) \arrow[dl, shift left, "D"] \arrow[dr, shift left, "I"] & \\
\mathrm{S}(\mathcal{M})^{op} \arrow[ur, shift left, "T"] \arrow[rr, shift right, "L"'] & & \mathrm{S}(\mathcal{M}) \arrow[ul, shift left, "J"] \arrow[ll, shift right, "R"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathrm{CL}(\mathbb{X}, \mathcal{M}) \arrow[dl, shift left, "D"] \arrow[dr, shift left, "I"] & \\
\mathrm{S}(\mathcal{M})^{op} \arrow[ur, shift left, "T"] \arrow[rr, shift right, "L"'] & & \mathrm{S}(\mathcal{M}) \arrow[ul, shift left, "J"] \arrow[ll, shift right, "R"']
\end{tikzcd}
$$

$$R(\mathcal{A}) = \{m \in \mathcal{M} \,|\, \forall a \in \mathcal{A} \;\; m \perp a\}$$

$$L(\mathcal{A}) = \{m \in \mathcal{M} \,|\, \forall a \in \mathcal{A} \;\; a \perp m\}$$

$$\begin{tikzcd}[column sep=huge, row sep=large]
\mathrm{S}(\mathcal{M})^{op} \arrow[rr, shift right, "L"'] \arrow[dr, shift left, "T"] \arrow[dd, shift left, "\gamma"] & & \mathrm{S}(\mathcal{M}) \arrow[ll, shift right, "R"'] \arrow[dl, shift left, "J"] \arrow[dd, shift left, "\gamma"] \\
& \mathrm{CL}(\mathbb{X}, \mathcal{M}) \arrow[ul, shift left, "D"] \arrow[ur, shift left, "I"] \arrow[dl, shift left, "D_{\mathcal{M}}"] \arrow[dr, shift left, "I_{\mathcal{M}}"] & \\
\mathrm{S}(\mathbb{X})^{op} \arrow[uu, shift left, "\theta"] \arrow[rr, shift left, "L_{\mathcal{M}}"] \arrow[ur, shift left, "T_{\mathcal{M}}"] & & \mathrm{S}(\mathbb{X}) \arrow[uu, shift left, "\theta"] \arrow[ll, shift left, "R_{\mathcal{M}}"] \arrow[ul, shift left, "J_{\mathcal{M}}"]
\end{tikzcd}$$

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\mathrm{S}(\mathcal{M})^{op} \arrow[rr, shift right, "L"'] \arrow[dr, shift left, "T"] \arrow[dd, shift left, "\gamma"] & & \mathrm{S}(\mathcal{M}) \arrow[ll, shift right, "R"'] \arrow[dl, shift left, "J"] \arrow[dd, shift left, "\gamma"] \\
& \mathrm{CL}(\mathbb{X}, \mathcal{M}) \arrow[ul, shift left, "D"] \arrow[ur, shift left, "I"] \arrow[dl, shift left, "D_{\mathcal{M}}"] \arrow[dr, shift left, "I_{\mathcal{M}}"] & \\
\mathrm{S}(\mathbb{X})^{op} \arrow[uu, shift left, "\theta"] \arrow[ur, shift left, "T_{\mathcal{M}}"] \arrow[rr, shift left, "L_{\mathcal{M}}"] & & \mathrm{S}(\mathbb{X}) \arrow[uu, shift left, "\theta"] \arrow[ul, shift left, "J_{\mathcal{M}}"] \arrow[ll, shift left, "R_{\mathcal{M}}"]
\end{tikzcd}
$$

$$\theta(\mathbb{A}) = \{m \in \mathcal{M} \mid \mathrm{codom}(m) \in \mathbb{A}\}$$

$$\gamma(\mathcal{A}) = \{A \,|\, (\forall m \in \mathcal{M}/A)\ m \in \mathcal{A}\}$$

$$
\begin{tikzcd}
\mathrm{S}(\mathcal{M})^{op} \arrow[rr, shift right, "L"'] \arrow[dr, shift left, "T"] \arrow[dd, shift left, "\gamma"] & & \mathrm{S}(\mathcal{M}) \arrow[ll, shift right, "R"'] \arrow[dl, shift left, "J"] \arrow[dd, shift left, "\gamma"] \\
& \mathrm{CL}(\mathbb{X}, \mathcal{M}) \arrow[ul, shift left, "D"] \arrow[ur, shift left, "I"] \arrow[dl, shift left, "D_{\mathcal{M}}"] \arrow[dr, shift left, "I_{\mathcal{M}}"] & \\
\mathrm{S}(\mathbb{X})^{op} \arrow[uu, shift left, "\theta"] \arrow[ur, shift left, "T_{\mathcal{M}}"] \arrow[rr, shift left, "L_{\mathcal{M}}"] & & \mathrm{S}(\mathbb{X}) \arrow[uu, shift left, "\theta"] \arrow[ul, shift left, "J_{\mathcal{M}}"] \arrow[ll, shift left, "R_{\mathcal{M}}"]
\end{tikzcd}
$$

$$\theta(\mathbb{A}) = \{m \in \mathcal{M} \mid \mathrm{codom}(m) \in \mathbb{A}\}$$

$$\gamma(\mathcal{A}) = \{A \,|\, (\forall m \in \mathcal{M}/A)\; m \in \mathcal{A}\}$$

$\mathcal{N} \subseteq \mathcal{M}$ $\qquad\qquad$ $T_{\mathcal{N}}(\mathbb{A}) = T\left(\theta(\mathbb{A}) \cap \mathcal{N}\right)$

$$\mathcal{D} = \{\delta_X \mid \delta_X = X \to X^2\}$$

$T_{\mathcal{D}}(\mathbb{A}^2) = \mathrm{reg}^{\mathbb{A}}$ and $J_{\mathcal{D}}(\mathbb{A}^2) = \mathrm{coreg}^{\mathbb{A}}$.

$$A \in \Delta(c) \Leftrightarrow A^2 \in D_{\mathcal{D}}$$
$$A \in \nabla(c) \Leftrightarrow A^2 \in I_{\mathcal{D}}$$

$\mathcal{D} = \{\delta_X \mid \delta_X = X \to X^2\}$

$T_{\mathcal{D}}(\mathbb{A}^2) = \mathrm{reg}^{\mathbb{A}}$ and $J_{\mathcal{D}}(\mathbb{A}^2) = \mathrm{coreg}^{\mathbb{A}}$.

$A \in \Delta(c) \Leftrightarrow A^2 \in D_{\mathcal{D}}$
$A \in \nabla(c) \Leftrightarrow A^2 \in I_{\mathcal{D}}$

When is $l(\mathbb{A})^2 = L_{\mathcal{D}}(\mathbb{A}^2) \cap \mathbb{X}^2$?

$$\mathcal{D} = \{\delta_X \mid \delta_X = X \to X^2\}$$

$T_{\mathcal{D}}(\mathbb{A}^2) = \mathrm{reg}^{\mathbb{A}}$ and $J_{\mathcal{D}}(\mathbb{A}^2) = \mathrm{coreg}^{\mathbb{A}}$.

$A \in \Delta(c) \Leftrightarrow A^2 \in D_{\mathcal{D}}$
$A \in \nabla(c) \Leftrightarrow A^2 \in I_{\mathcal{D}}$

When is $l(\mathbb{A})^2 = L_{\mathcal{D}}(\mathbb{A}^2) \cap \mathbb{X}^2$?

$X \parallel Y$ if $(\forall f : X \to Y)$ $f(X)$ is preterminal.

$\mathcal{D} = \{\delta_X \mid \delta_X = X \to X^2\}$

$T_{\mathcal{D}}(\mathbb{A}^2) = \mathrm{reg}^{\mathbb{A}}$ and $J_{\mathcal{D}}(\mathbb{A}^2) = \mathrm{coreg}^{\mathbb{A}}$.

$A \in \Delta(c) \Leftrightarrow A^2 \in D_{\mathcal{D}}$
$A \in \nabla(c) \Leftrightarrow A^2 \in I_{\mathcal{D}}$

When is $l(\mathbb{A})^2 = L_{\mathcal{D}}(\mathbb{A}^2) \cap \mathbb{X}^2$?

$X \parallel Y$ if $(\forall f : X \to Y)$ $f(X)$ is preterminal.

We need to compare $X \parallel Y$ with $\delta_X \perp \delta_Y$.

If $\mathcal{E}$ is closed under the formation of squares, then $\delta_X \perp \delta_Y \Rightarrow X \parallel Y$.

If $\mathcal{E}$ is closed under the formation of squares, then
$\delta_X \perp \delta_Y \Rightarrow X \parallel Y$.

If for all $X \in \mathbb{X}$
$1_{X^2} = \bigvee\{1_X \times x \mid x$ is a point of $X\} \vee \bigvee\{x \times 1_X \mid x$ is a point of $X\}$,
then $X \parallel Y \Rightarrow \delta_X \perp \delta_Y$.

If $\mathcal{E}$ is closed under the formation of squares, then
$\delta_X \perp \delta_Y \Rightarrow X \parallel Y$.

If for all $X \in \mathbb{X}$
$1_{X^2} = \bigvee\{1_X \times x \mid x$ is a point of $X\} \vee \bigvee\{x \times 1_X \mid x$ is a point of $X\}$,
then $X \parallel Y \Rightarrow \delta_X \perp \delta_Y$.

Under the conditions above
$r(\mathbb{A}) = \Delta(\mathrm{coreg}^{\mathbb{A}})$ and $l(\mathbb{A}) = \nabla(\mathrm{reg}^{\mathbb{A}})$.

Another Example

Top with the (surjections, embeddings) factorization.

$\mathcal{P} = \{x \mid x : 1 \to X\}$

Another Example

Top with the (surjections, embeddings) factorization.

$\mathcal{P} = \{x \mid x : 1 \to X\}$

$\mathrm{fib}^{\mathbb{A}} = T_{\mathcal{P}}(\mathbb{A})$ and $\mathrm{fib}^{\mathcal{A}} = T(\mathcal{A})$ fibre closure

$\mathrm{cofib}^{\mathbb{A}} = J_{\mathcal{P}}(\mathbb{A})$ and $\mathrm{cofib}^{\mathcal{A}} = J(\mathcal{A})$ cofibre closure

Another Example

Top with the (surjections, embeddings) factorization.

$\mathcal{P} = \{x \mid x : 1 \to X\}$

$\mathrm{fib}^{\mathbb{A}} = T_{\mathcal{P}}(\mathbb{A})$ and $\mathrm{fib}^{\mathcal{A}} = T(\mathcal{A})$ fibre closure

$\mathrm{cofib}^{\mathbb{A}} = J_{\mathcal{P}}(\mathbb{A})$ and $\mathrm{cofib}^{\mathcal{A}} = J(\mathcal{A})$ cofibre closure

k is the fibre closure of the class of the closed points.

$I_{\mathcal{P}}(k)$ is the class of the indiscrete spaces.

$D_{\mathcal{P}}(k) = \mathsf{Top}_1$

$D_{\mathcal{P}}(\mathrm{cofib}^{I_{\mathcal{P}}(k)}) = \mathsf{Top}_0$

$\mathsf{Top}_2 = D_{\mathcal{P}}(c)$ for c such that

$c_X(M) = \{x \in X \mid \forall U, V \in \mathcal{T}\ (x \in U$ and $M \subseteq V \Rightarrow U \cap V \neq \emptyset\}$

Another Example

Top with the (surjections, embeddings) factorization.

$\mathcal{P} = \{x \mid x : 1 \to X\}$

$\mathrm{fib}^{\mathbb{A}} = T_{\mathcal{P}}(\mathbb{A})$ and $\mathrm{fib}^{\mathcal{A}} = T(\mathcal{A})$ fibre closure

$\mathrm{cofib}^{\mathbb{A}} = J_{\mathcal{P}}(\mathbb{A})$ and $\mathrm{cofib}^{\mathcal{A}} = J(\mathcal{A})$ cofibre closure

k is the fibre closure of the class of the closed points.

$I_{\mathcal{P}}(k)$ is the class of the indiscrete spaces.

$D_{\mathcal{P}}(k) = \mathsf{Top}_1$

$D_{\mathcal{P}}(\mathrm{cofib}^{I_{\mathcal{P}}(k)}) = \mathsf{Top}_0$

$\mathsf{Top}_2 = D_{\mathcal{P}}(c)$ for c such that

$c_X(M) = \{x \in X \mid \forall U, V \in \mathcal{T}\ (x \in U$ and $M \subseteq V \Rightarrow U \cap V \neq \emptyset\}$

$R_{\mathcal{P}}(\mathbb{A}) = r(\mathbb{A})$ and $L_{\mathcal{P}}(\mathbb{A}) = l(\mathbb{A})$

The same Example

$\mathsf{Top}_\star$ - pointed topological spaces.

$\mathcal{P}/\mathsf{Top} \simeq \mathsf{Top}_\star$ and $S(\mathcal{P}) \simeq S(\mathsf{Top}_\star)$

The same Example

$\mathsf{Top}_\star$ - pointed topological spaces.

$\mathcal{P}/\mathsf{Top} \simeq \mathsf{Top}_\star$ and $S(\mathcal{P}) \simeq S(\mathsf{Top}_\star)$

$\mathcal{A} \subseteq \mathcal{P}$

$\mathrm{fib}^{\mathcal{A}}$ in Top is equal to $\mathrm{norm}^{\mathcal{A}}$ in $\mathsf{Top}_\star$.

The same Example

$\mathsf{Top}_\star$ - pointed topological spaces.

$\mathcal{P}/\mathsf{Top} \simeq \mathsf{Top}_\star$ and $S(\mathcal{P}) \simeq S(\mathsf{Top}_\star)$

$\mathcal{A} \subseteq \mathcal{P}$

$\mathrm{fib}^{\mathcal{A}}$ in Top is equal to $\mathrm{norm}^{\mathcal{A}}$ in $\mathsf{Top}_\star$.

k is a normal closure.