

On (co)normal closure operators

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Joint work with Maria Manuel Clementino

Given a class of groups $\mathbb{A} \subseteq \text{Grp}$, the normal closure induced by \mathbb{A} is given by

$$\text{norm}_{G}^{\mathbb{A}}(H) := \bigcap \{N \mid H \subseteq N \triangleleft G, G/N \subseteq A \in \mathbb{A}\} = \bigcap \{f^{-1}(0) \mid f : G \rightarrow A \in \mathbb{A}, f(H) = 0\}.$$

It is easy to see that the normal closure can be defined in any category with a 0-object and an \mathcal{M} -right factorization, where \mathcal{M} contains all normal monomorphisms.

It is patent that the constructions of the normal and the regular closure are very similar. Accordingly, we will define the conormal closure - in parallel with the coregular closure - and, using a unifying setting, we will generalize results obtained for regular/coregular closures in [CT].

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$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow e & \nearrow m \\ & f(X) & \end{array}$$

$$\begin{array}{ccccc} M & \xrightarrow{m} & X & \xrightarrow{f} & Y \\ & \searrow & & \nearrow f(m) & \\ & f(M) & & & \end{array}$$

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$$\begin{array}{ccc}
 f^{-1}(M) & \longrightarrow & M \\
 \downarrow f^{-1}(m) & & \downarrow m \\
 X & \xrightarrow{f} & Y
 \end{array}$$

A closure operator c on \mathbb{X} w.r.t. \mathcal{M} is a family of functions $(c_X : \mathcal{M}/X \rightarrow \mathcal{M}/X)_{X \in \mathbb{X}}$:

1. $m \leq c_X(m)$;
2. if $m \leq m'$ then $c_X(m) \leq c_X(m')$;
3. $f(c_X(m)) \leq c_Y(f(m))$ for $f : X \rightarrow Y$, $m \in \mathcal{M}/X$.

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$c_X(f^{-1}(m)) \leq f^{-1}(c_Y(m))$ for $f : X \rightarrow Y$, $m \in \mathcal{M}/Y$

c is idempotent if $c(c(m)) = c(m)$ for all $m \in \mathcal{M}$.

c is weakly hereditary if $c(j_m) = 1_{cM}$ for all $m \in \mathcal{M}$ with $c(m) \cdot j_m = m$.

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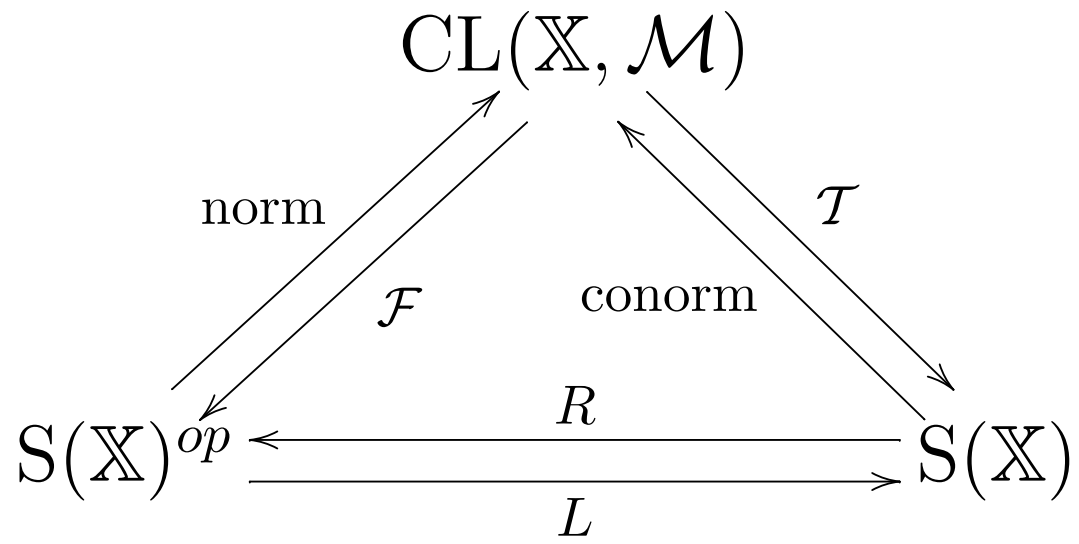
$$\text{S}(\mathbb{X})^{op} \begin{array}{c} \xrightarrow{\text{norm}} \\ \xleftarrow{\mathcal{F}} \end{array} \text{CL}(\mathbb{X}, \mathcal{M})$$

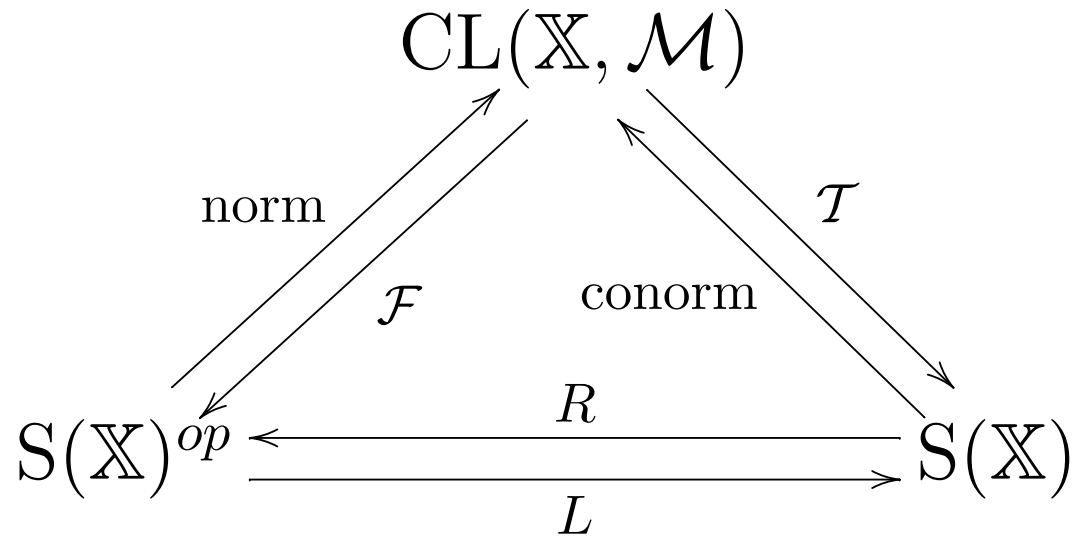
$$\text{CL}(\mathbb{X}, \mathcal{M}) \begin{array}{c} \xrightarrow{\mathcal{T}} \\ \xleftarrow{\text{conorm}} \end{array} \text{S}(\mathbb{X})$$

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$$\mathcal{T}(c) = \{X \mid c_X(0) = 1_X\}$$

$$\text{conorm}_X^{\mathbb{A}}(m) = m \vee \bigvee \{f(1_A) \mid f : A \rightarrow X, A \in \mathbb{A}, f(0) \leq m\}$$





$$R(\mathbb{A}) = \{X \mid (\forall f : X \rightarrow A, A \in \mathbb{A}) f(X) = 0\}$$

$$L(\mathbb{A}) = \{X \mid (\forall f : A \rightarrow X, A \in \mathbb{A}) f(A) = 0\}$$

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$$\text{coreg}_X^{\mathbb{A}}(m) = m \vee \bigvee \{f(1_{A^2}) \mid f : A^2 \rightarrow X, A \in \mathbb{A}, f(\delta_A) \leq m\}$$

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$$r(\mathbb{A}) = \{X \mid (\forall f : X \rightarrow A, A \in \mathbb{A}) f(X) \text{ is preterminal}\}$$

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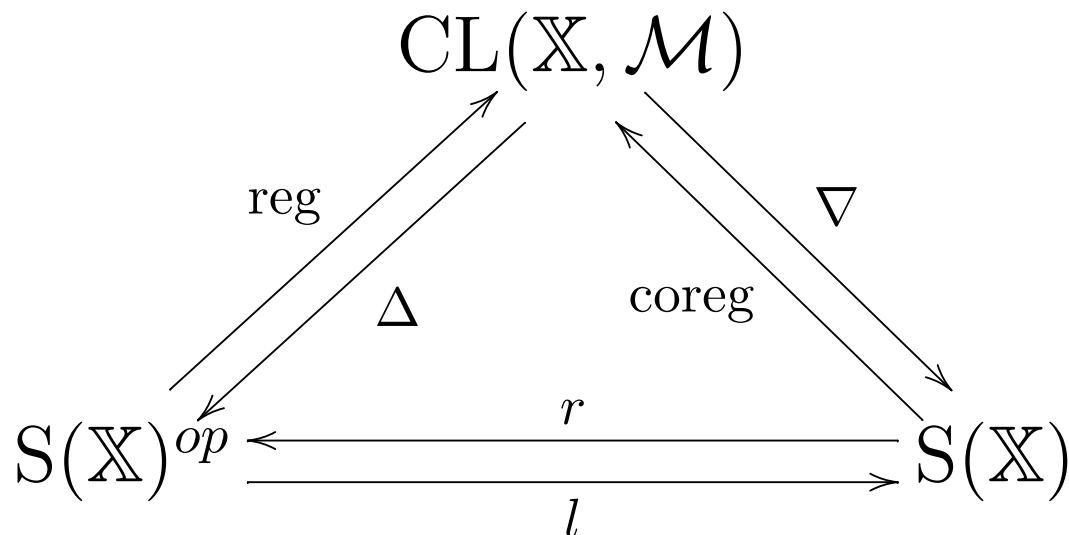
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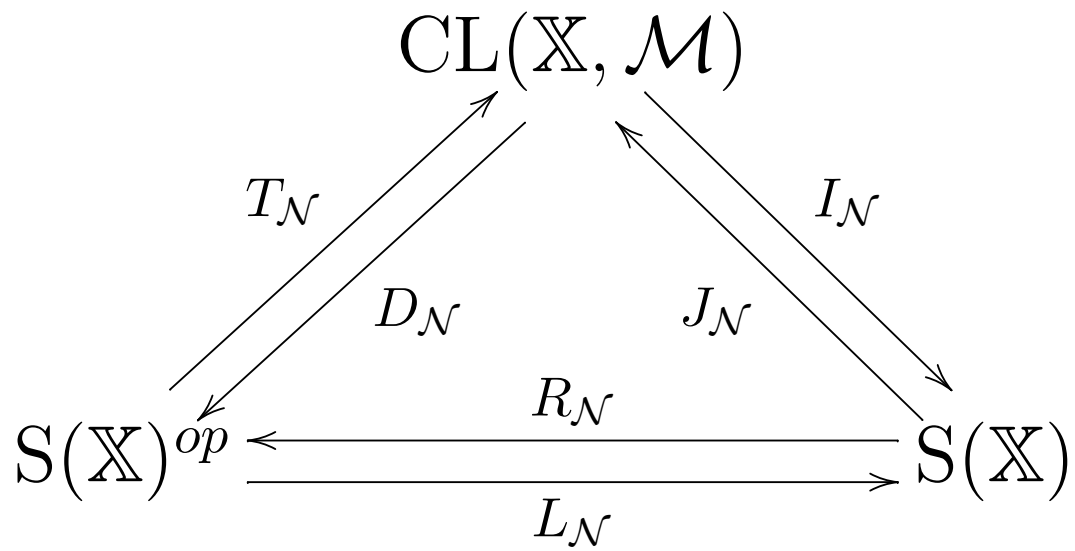
A point of X is $(x : 1 \rightarrow X) \in \mathcal{M}$ with 1 the terminal object.

$$1_X = \bigvee \{x \mid x \text{ is a point of } X\} \text{ for all } X \in \mathbb{X}$$

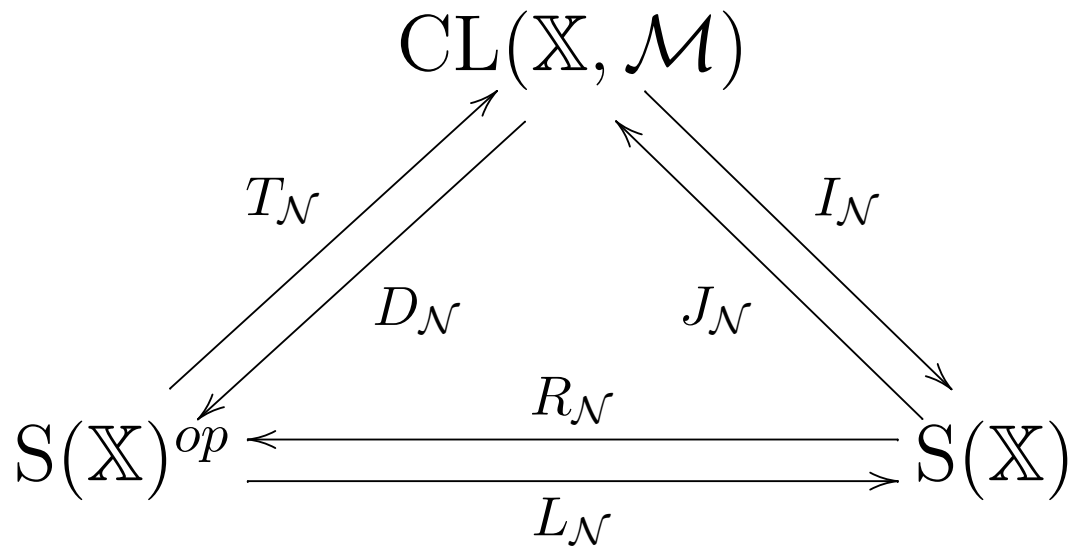
\mathcal{E} is closed under the formation of squares.



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$$D_{\mathcal{N}}(c) = \{X \mid \forall n \in \mathcal{N}/X \ c(n) = n\}$$

$$T_{\mathcal{N}}(\mathbb{A})_X(m) = \bigwedge \{f^{-1}(n) \mid f : X \rightarrow A \in \mathbb{A}, n \in \mathcal{N}/A, m \leq f^{-1}(n)\}$$

$$I_{\mathcal{N}}(c) = \{X \mid \forall n \in \mathcal{N}/X \ c(n) = 1_X\}$$

$$J_{\mathcal{N}}(\mathbb{A})_X(m) = m \vee \bigvee \{f(1_A) \mid f : A \rightarrow X, A \in \mathbb{A}, (\exists n \in \mathcal{N}) f(n) \leq m\}$$

Examples

$$\mathcal{Z} = \{0_X \mid 0_X : 0 \rightarrow X\}$$

$$T_{\mathcal{Z}} = \text{norm} \text{ and } J_{\mathcal{Z}} = \text{conorm.}$$

$$D_{\mathcal{Z}} = \mathcal{F} \text{ and } I_{\mathcal{Z}} = \mathcal{T}.$$

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$$\mathcal{D} = \{\delta_X \mid \delta_X : X \rightarrow X^2\}$$

$$\mathbb{A} \subseteq \mathbb{X} \quad \mathbb{A}^2 = \{A^2 \mid A \in \mathbb{A}\}$$

$$T_{\mathcal{D}}(\mathbb{A}^2) = \text{reg}^{\mathbb{A}} \text{ and } J_{\mathcal{D}}(\mathbb{A}^2) = \text{coreg}^{\mathbb{A}}.$$

$$A \in \Delta(c) \Leftrightarrow A^2 \in D_{\mathcal{D}}$$

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$\mathcal{A} = DT(\mathcal{A})$ iff and only if \mathbb{A} is an $iso\mathbb{X}$ -reflective subcategory of \mathcal{M}/\mathbb{X} .

A Torsion Theory?

Two morphisms $m, n \in \mathcal{M}$ are **orthogonal** $m \perp n$ if

$f(m) \leq n \Rightarrow f(1_X) \leq n$ for $f : X \rightarrow Y$, $m \in \mathcal{M}/X$, $n \in \mathcal{M}/Y$.

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A 'torsion theory' in \mathcal{M} is a pair $(\mathcal{A}, \mathcal{B})$ such that:

1. for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, $a \perp b$;
2. for every $m \in \mathcal{M}$, there is $a \in \mathcal{A}$, $b \in \mathcal{B}$ such that $m = b \cdot a$.

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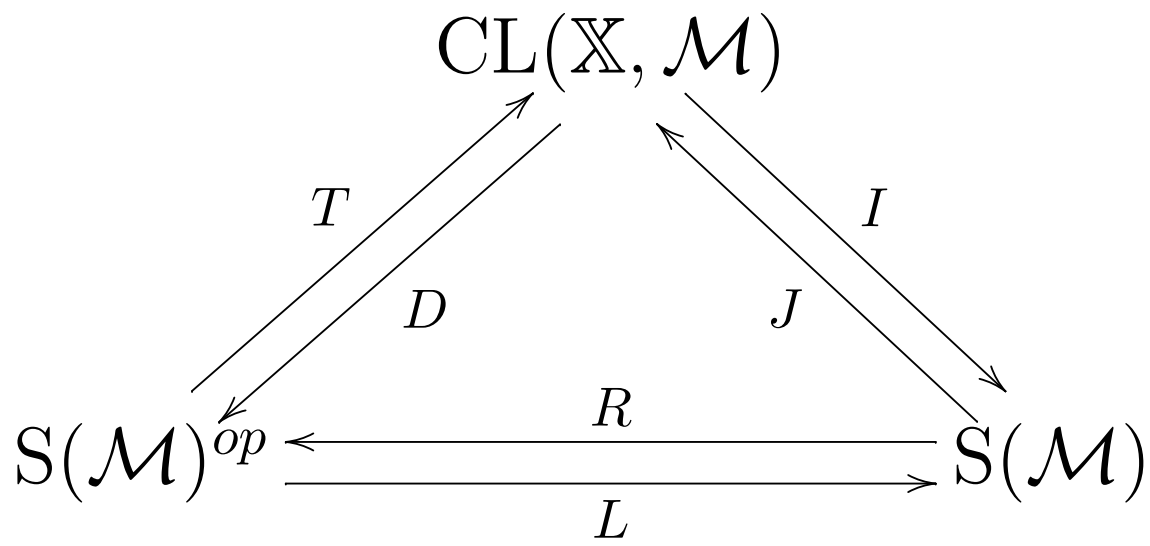
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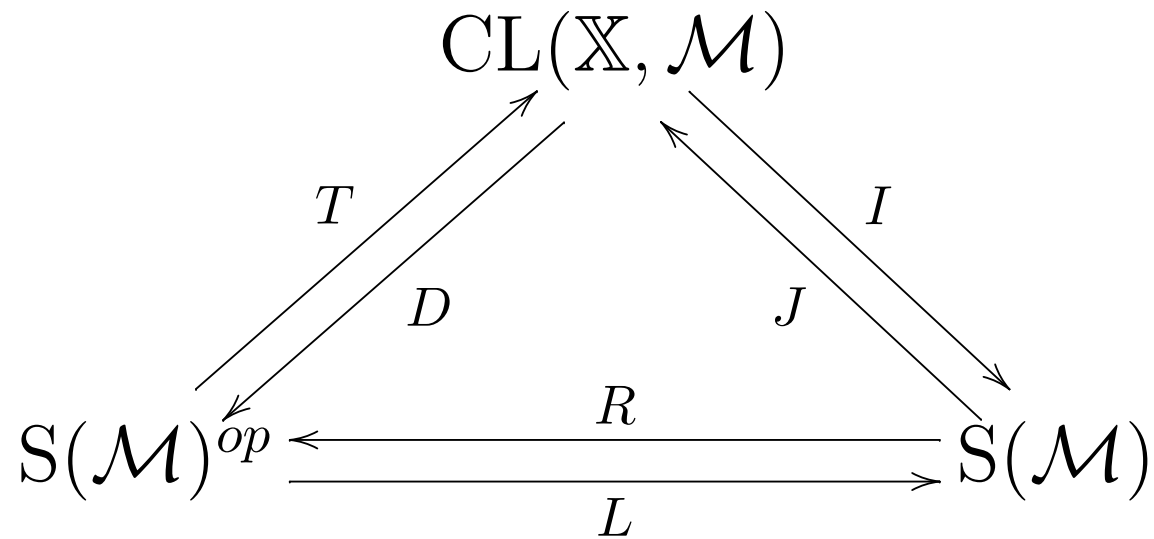
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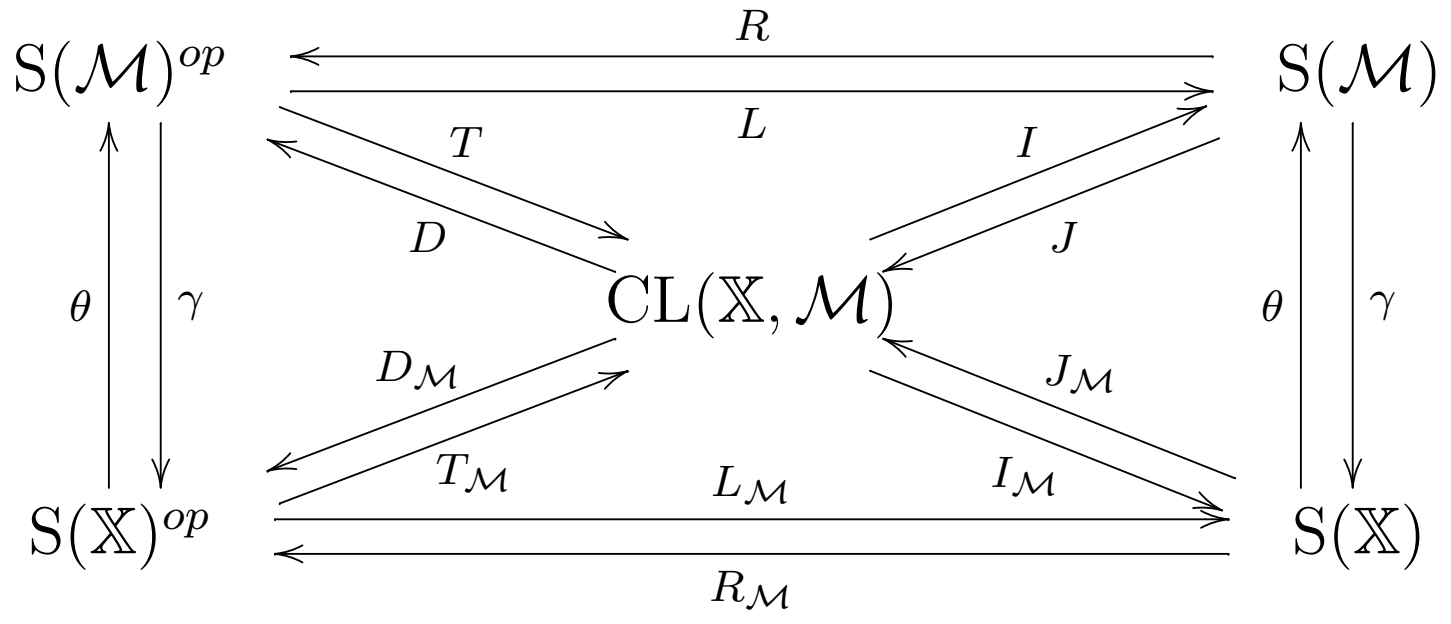
The pair $(\mathcal{A}, \mathcal{B})$ is a torsion theory if and only if there is an idempotent weakly hereditary closure operator c such that $\mathcal{A} = I(c)$ and $\mathcal{B} = D(c)$.

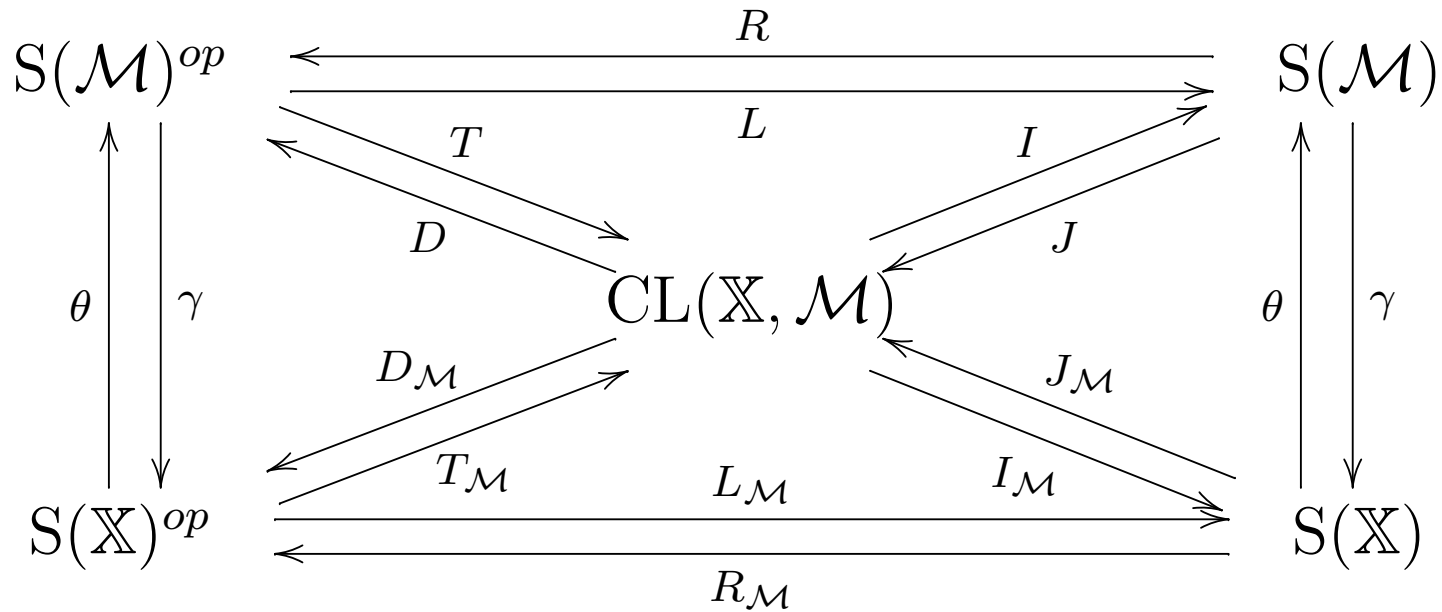




$$R(\mathcal{A}) = \{m \in \mathcal{M} \mid \forall a \in \mathcal{A} \ m \perp a\}$$

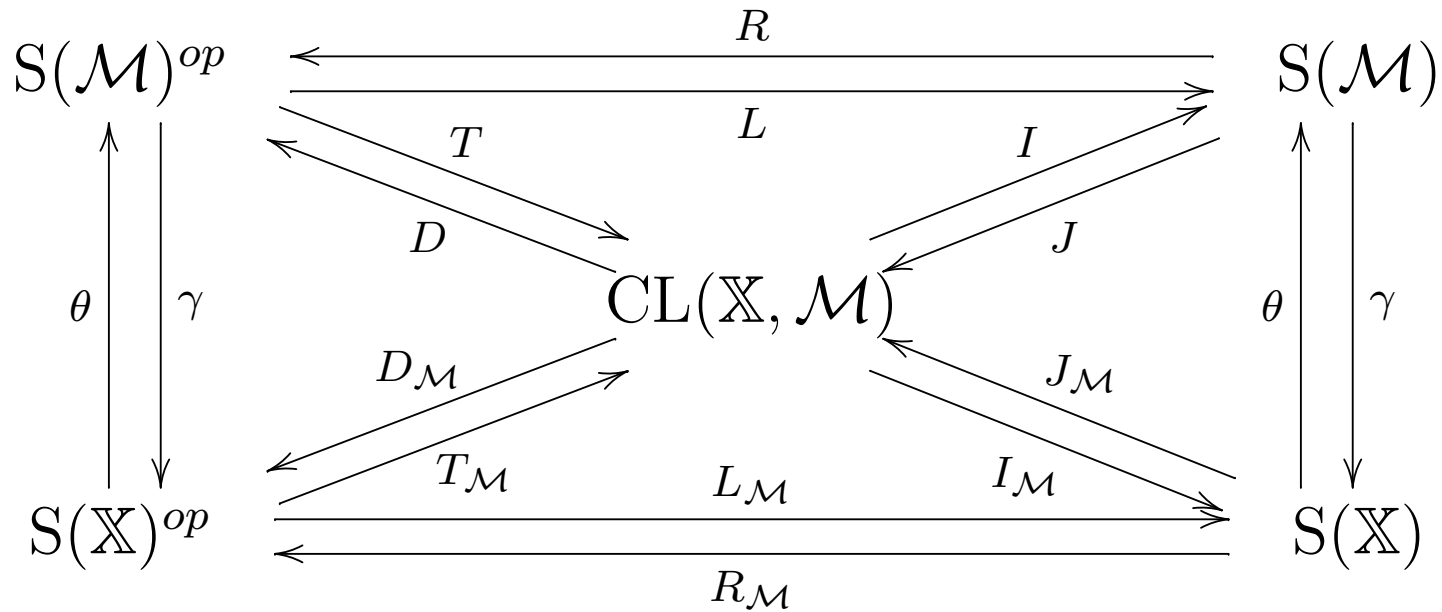
$$L(\mathcal{A}) = \{m \in \mathcal{M} \mid \forall a \in \mathcal{A} \ a \perp m\}$$





$$\theta(\mathbb{A}) = \{m \in \mathcal{M} \mid \text{codom}(m) \in \mathbb{A}\}$$

$$\gamma(\mathcal{A}) = \{A \mid (\forall m \in \mathcal{M}/A) m \in \mathcal{A}\}$$



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$$\mathcal{N} \subseteq \mathcal{M}$$

$$T_{\mathcal{N}}(\mathbb{A}) = T(\theta(\mathbb{A}) \cap \mathcal{N})$$

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When is $l(\mathbb{A})^2 = L_{\mathcal{D}}(\mathbb{A}^2) \cap \mathbb{X}^2$?

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When is $l(\mathbb{A})^2 = L_{\mathcal{D}}(\mathbb{A}^2) \cap \mathbb{X}^2$?

$X \parallel Y$ if $(\forall f : X \rightarrow Y) f(X)$ is preterminal.

We need to compare $X \parallel Y$ with $\delta_X \perp \delta_Y$.

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Under the conditions above

$r(\mathbb{A}) = \Delta(\text{coreg}^{\mathbb{A}})$ and $l(\mathbb{A}) = \nabla(\text{reg}^{\mathbb{A}})$.

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Top with the (surjections, embeddings) factorization.

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fibre closure

cofibre closure

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cofibre closure

k is the fibre closure of the class of the closed points.

$I_{\mathcal{P}}(k)$ is the class of the indiscrete spaces.

$$D_{\mathcal{P}}(k) = \text{Top}_1$$

$$D_{\mathcal{P}}(\text{cofib}^{I_{\mathcal{P}}(k)}) = \text{Top}_0$$

$\text{Top}_2 = D_{\mathcal{P}}(c)$ for c such that

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$$R_{\mathcal{P}}(\mathbb{A}) = r(\mathbb{A}) \text{ and } L_{\mathcal{P}}(\mathbb{A}) = l(\mathbb{A})$$

The same Example

Top_* - pointed topological spaces.

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