

Sequential Approach Spaces

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Approach spaces were introduced by R. Lowen as a natural generalisation of both topological and metric spaces. Therefore many concepts in approach theory are motivated by well-established notions in topology such as compactness or soberness.

With no doubt, sequential spaces form an important classe of topological spaces, however, so far appears no study of sequential approach spaces in the literature. It is the goal of this talk to fill this gap. We propose a definition of sequential approach spaces and some of its properties are studied.

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$$\delta(x, A) \leq \delta(x, A^{(\varepsilon)}) + \varepsilon$$

$$A^{(\varepsilon)} = \{x \mid \delta(x, A) \leq \varepsilon\}$$

Fréchet space

$$\{\delta \in [0, +\infty]^{X \times PX}\} \begin{array}{c} \xrightarrow{c} \\ \xleftarrow{s} \end{array} \{a \in [0, +\infty]^{SX \times X}\}$$

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$$\mathcal{C}(\delta)((x_n), x) := \sup_{(y_n) \preceq (x_n)} \delta(x, \{y_n \mid n \in \mathbb{N}\})$$

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δ is a Fréchet Approach space if $\delta = \mathcal{S}(\mathcal{C}(\delta))$

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\mathcal{S}^* is a Fréchet Approach distance.

In general this construction gives a reflection.

Galois connection

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$\mathcal{R}(a)$ is a regular function frame on X .

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(X, ρ) is a sequential approach space if $\rho = \mathcal{RC}(\rho)$

(X, δ) is a sequential approach space if $\delta = \mathcal{S}^*$

Examples

Topological sequential spaces

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Metric spaces

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Topological sequential spaces

Metric spaces

$$\mathbb{P} = ([0, \infty], \delta)$$

$$\delta(x, A) := x - \sup A$$

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$$a^{\alpha+1} = a \cdot Sa^\alpha \quad \text{i.e.} \quad a^{\alpha+1}(\mathfrak{z}, x) = \inf_{(y_n)} Sa^\alpha(\mathfrak{z}, (y_n)) + a((y_n), x)$$

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Lemma $S^\alpha(x, A) = \inf_{\mathfrak{z} \in S^\alpha A} a(\mathfrak{z}, x)$

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$$a((x_k), x) \leq \inf_n \sup_{k \geq n} a(\dot{x}_k, y_k) + a((y_n), x)$$

Some results

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- Sequential approach spaces is a topological category.

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