

Totally bounded metric spaces and the Axiom of Choice

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A metric space is *totally bounded* if it has a finite ϵ -net for every $\epsilon > 0$. One can prove that a metric space is topologically equivalent to a totally bounded metric space if and only if it has a countable ϵ -net for every $\epsilon > 0$ if and only if it is a Lindelöf space if and only if it is second countable if and only if it is separable if and only if ...

These equivalences do not remain valid in **ZF** (*Zermelo-Fraenkel set theory without the Axiom of Choice*). In this talk we will discuss the set-theoretic status of these equivalences as well as of the other results related with totally boundness.

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CUC – Countable Union Condition.

The countable union of countable sets is countable.

$$\mathbf{CC}(2^{\aleph_0}) \Rightarrow \mathbf{CUC} \Rightarrow \mathbf{CC}(\aleph_0)$$

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- (vii) Weakly Lindelöf – every open cover has a countable open refinement;
- (viii) Super Second Countable – every base for the open sets contains a countable base;

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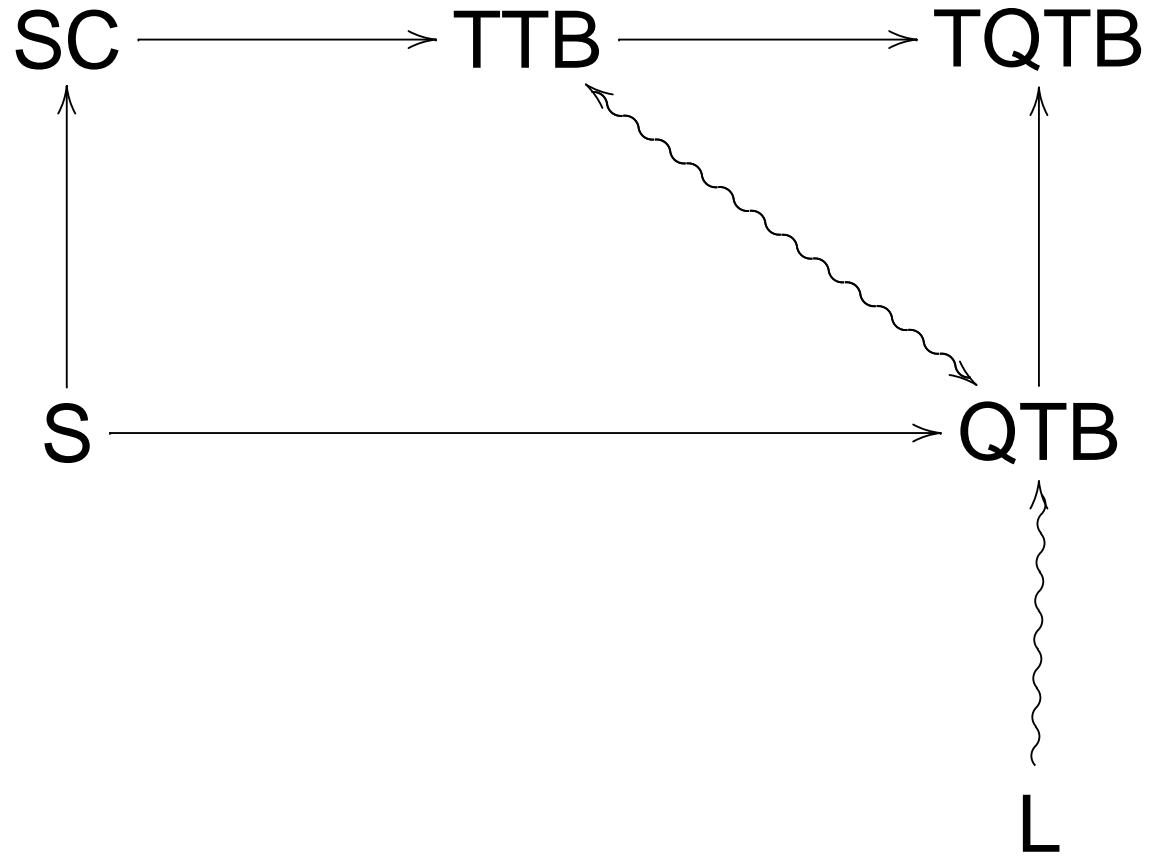
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(x) Topologically Quasi Totally Bounded (TQTB).



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$$(5) \quad (\mathbf{SC} \Rightarrow \text{QTB}) \Leftrightarrow \mathbf{CC}(\mathbb{R}).$$

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Theorem. Every Lindelöf pseudometric space is Quasi Totally Bounded if and only if \mathbf{CC} holds or $\mathbf{CC}(\mathbb{R})$ fails.

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