Totally bounded metric spaces and the Axiom of Choice

Gonçalo Gutierres – CMUC/Universidade de Coimbra

A metric space is *totally bounded* if it has a finite ϵ-net for every $\epsilon > 0$. One can prove that a metric space is topologically equivalent to a totally bounded metric space if and only if it has a countable ϵ-net for every $\epsilon > 0$ if and only if it is a Lindelöf space if and only if it is second countable if and only if it is separable if and only if ...

These equivalences do not remain valid in ZF (*Zermelo-Fraenkel set theory without the Axiom of Choice*). In this talk we will discuss the set-theoretic status of these equivalences as well as of the other results related with totally boundness.

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$\mathbf{CUC}$ – Countable Union Condition.
The countable union of countable sets is countable.

$$\mathbf{CC}(2^{\aleph_0}) \;\Rightarrow\; \mathbf{CUC} \;\Rightarrow \mathbf{CC}(\aleph_0)$$

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(viii) Super Second Countable – every base for the open sets contains a countable base;

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(x) Topologically Quasi Totally Bounded (TQTB).

$$
\begin{tikzcd}
SC \arrow[r] & TTB \arrow[r] \arrow[dr, squiggly, leftrightarrow] & TQTB \\
S \arrow[u] \arrow[rr] & & QTB \arrow[u] \\
& & L \arrow[u, squiggly]
\end{tikzcd}
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(4) (TB$\Rightarrow$ SC) $\Rightarrow$ **CC**(fin).

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(3) (QTB $\Rightarrow$ L) $\Rightarrow$ $\mathbf{CC}(\mathbb{R})$+ $\mathbf{CUC}$.

(4) (TB$\Rightarrow$ SC) $\Rightarrow$ $\mathbf{CC}$(fin).

(5) (SC $\Rightarrow$ QTB) $\Leftrightarrow$ $\mathbf{CC}(\mathbb{R})$.

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Theorem. Every Lindelöf pseudometric space is Quasi Totally Bounded if and only if $\mathbf{CC}$ holds or $\mathbf{CC}(\mathbb{R})$ fails.

References

H. L. Bentley and H. Herrlich, *Countable choice and pseudometric spaces*, Topology Appl. **85** (1997).

G. Gutierres, *On first and second countable spaces and the axiom of choice*, Topology Appl. **143** (2004).

H. Herrlich, *Products of Lindelöf T_2-spaces are Lindelöf – in some models of* ZF, Com. M. U. Carolinae **43** (2002).

P. Howard and J. E. Rubin, *Consequences of the Axiom of Choice*, American Mathematical Society, 1998. `http://consequences.emich.edu/conseq.htm`.

K. Keremedis, *Disasters in topology without the axiom of choice*, Arch. Math. Logic **40** (2001).