The Axiom of Countable Choice in Topology

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Introduction

> *Topologists not only employed the Axiom [of Choice] with ease but often took it for granted. None exhibited the qualms of conscience which affected van der Waerden in Algebra – despite the fact several of the topologists were students of Luzin, an ardent constructivist. Perhaps topologists sensed that it was inappropriate, when engaged in so abstract an enterprise, to insist on constructive methods. Perhaps the enchanting landscape of abstraction and generalization freed them from any lingering constructivist scruples. Whatever the reason, they did not seriously entertain the question of what general topology might become in the absence of the Axiom. Like so many mathematicians before and afterward, the use of arbitrary choices became a second nature to them – if not, indeed, a reflex.*
>
> Gregory H. Moore [36, p.242]

The Axiom of Choice is certainly one of the most discussed subjects in the History of Mathematics. It was formally introduced by E. Zermelo [48] in 1904 to prove Cantor's Well Ordering Principle [5].

Until that date, the Axiom of Choice was implicitly used to prove many other results. Even several of the most elementary properties of the real numbers, already known at the time, require the use of some form of choice. Some of these properties will be investigated in the second chapter of this work.

After many discussions, about his new axiom and his proof of the Well Ordering Principle, Zermelo did a first attempt to axiomatize Set Theory [49, 50] and he included the Axiom of Choice. Since the Axiom of Choice was not universally accepted, two versions of that theory are usually considered, one with and the other without the Axiom of Choice. Due to additions and improvements introduced by A. Fraenkel this theory is called *Zermelo-Fraenkel Set Theory* and denoted by ZF, without the Axiom of Choice, or by ZFC, with the Axiom of Choice. There are several axiomatizations of ZF, but for our purpose the differences do not matter. To be more precise, I use the axiomatization of T. J. Jech [31]. With only small modifications of the axioms it is possible to allow the existence of atoms in the theory. The *Zermelo-Fraenkel Set Theory with atoms*, but without the Axiom of Choice, is denoted by ZFA. Here we will work in ZF, even though in general the results are also true in ZFA.

Today, the Axiom of Choice is used without hesitation in areas such as algebra, analysis, logic, set theory and topology. It is, in fact, known that many results of classical analysis are based in the *Axiom of Dependent Choice*, and that many results of algebra rely on the systematic use of the *Boolean Prime Ideal Theorem*. Both of these principles are properly weaker than the Axiom of Choice.

Due to its close relationship with Set Theory, Topology was always an area deeply affected by the use of the Axiom of Choice. Even though nowadays no one questions the legitimacy of its use in General Topology, it is interesting to investigate in which proofs AC is used and which theorems cannot be proved without its help.

Since F. Hausdorff introduced the notion of topological space, topologistes use the Axiom of Choice freely. Even before that, Maurice Fréchet had worked with spaces defined by sequences and implicitly he had used the Axiom of Dependent Choice strongly. Following, indirectly, his steps the fourth chapter of this text is devoted to the investigation of the role of AC in the definition of certain classes of topological spaces by means of limits. As well as Fréchet also Hausdorff, and after them others as W. Sierpiński, K. Kuratowski, P. Alexandroff or P. Urysohn, had used the Axiom to obtain the first results about metric and topological spaces. Many of the results initially proved by these Mathematicians require only the Axiom of Countable Choice, in particular results related to separable, Lindelöf or second countable spaces. Only much later (with exception of Sierpiński [41, 42]), the kind of choice used for this type of results became the object of study (e.g. [15, 2, 33]). Such problems will be analysed in Section 3.1.

Probably the best known topological theorem equivalent to the Axiom of Choice is Tychonoffs Compactness Theorem ([45]). This equivalence was proved by J. L. Kelley [32] in 1950 and represents the beginning of the study, in a systematic way, of the influence of the Axiom of Choice in General Topology. In fact what A. Tychonoff proved originally was that the product of any number of copies of the closed interval [0,1] is compact. Although his proof can be used in the general case. It is interesting to notice that Tychonoff's original result is equivalent to the Boolean Prime Ideal Theorem ([38, 37]) which, as it was already mentioned, is properly weaker than AC.

In the first part of the last century, some Mathematicians have pointed out the use of the Axiom of Choice in the proof of several results, mainly in Set Theory. But they had no tools that allowed to prove that those results are not provable in ZF. This problem was partial solved by the Fraenkel-Mostowski method of generating models of ZFA and completely solved in 1963 by P. J. Cohen [7] who created the *forcing* method of building models of ZF. This method made it possible to investigate, in ZF, the relations between the different weak forms of AC. These new possibilites created an interest in the question which precise choice principle is equivalent to some specific topological theorem.

The independence proofs in this dissertation are based on known properties of various models of ZF. Most of these models and their properties are described in *Consequences of the Axiom of Choice* [29] from P. Howard and J. E. Rubin and its on-line version [28]. (The latter is being updated and the search is easier.)

The goal of this work is to investigate the role of choice principles in the proof of several topological results. In particular an attempt is made to find for each topological theorem discussed an equivalent choice principle.

A considerable part of the research here produced is related with "countability concepts". Among others, the classes of: separable, Lindelöf, sequentially compact, first countable, second countable, Fréchet-Urysohn, sequential spaces are studied. For this reason the Axiom of Countable Choice is most of the times sufficient to prove results concerning such classes.

From now on, we will discuss more in detail the subject of this work. In the first chapter concepts and results are introduced, both from Set Theory and Topology, that will be used in the following chapters. This is necessary (even for familiar concepts) in view of the fact that some topological concepts can be described in various ways that are equivalent in ZFC, but often fail to be equivalent in ZF. Also several results that hold in ZF are presented.

The second chapter is devoted to the study of topological properties of the reals. It is shown that many familiar properties of the reals depend on choice principles. The chapter is divided in two parts. In the first part it is shown that $\mathbb{R}$ is a Fréchet-Urysohn space if and only if the Axiom of Countable Choice holds for families of subsets of the reals ($\mathbf{CC}(\mathbb{R})$), and moreover that these conditions are equivalent to several other familiar topological statements about the real numbers. In the second part the question under which conditions the reals form a sequential space is being investigated. Among other things, it is shown that this condition is properly implied by $\mathbf{CC}(\mathbb{R})$.

The third chapter is entitled *Countable Properties* and in there are studied separable, Lindelöf and second countable spaces. In the first of its four sections is studied the equivalence between these three properties in the class of metric spaces. In the second, it is show that a well-known characterization of second countable spaces is not provable in ZF. In the last two sections, the attention is centered in the stability of the above three properties for products respectively countable products.

In the fourth chapter, properties concerning limits of sequences and ultrafilters are investigated in detail. We start by generalizing to the class of first countable spaces results about Fréchet-Urysohn and sequencial spaces, which were already studied in the second chapter for $\mathbb{R}$. In the third section it is made a parallel study to this one for the spaces defined by limits of ultrafilters. Let us point out that in ZFC all topological spaces can be characterized by means of ultrafilters, whereas this is not possible in ZF. In the second section relations between the notions of compact, sequentially compact and countable compact spaces are studied.

In the fifth chapter the question under which conditions *every metric space has a unique completion* is investigated. It is shown that the answer heavily depends on the definition of the concepts complete and dense. Finally, the third section of the chapter presents few results about products of metric and complete metric spaces.

In the sixth and last chapter several definitions and characterizations of first countable spaces are compared. It is provided different characterizations of first countable spaces in ZFC. These are divided in two groups. In ZF, the first three reproduce the original idea of first countable at three different levels, and any of them might be considered as a definition. The other six characterizations are not desirable as definitions due to their complexity. We will see in what conditions the characterizations naturally related are equivalent, and also if, and when, some special topological spaces satisfy them. Once again is given particular relevance to the space of the reals.

The three last chapters of this work (with exception of the Sections 4.2 and 5.3.) and the Sections 2.2, 3.2 and 3.3 are considered original. There are a few exceptions to this rule, pointed out in appropriate places.

Chapter 1

Definitions and auxiliar results

1.1 Set Theory

(...)

In some definitions we will use the word *family* in stead of the word *set*. This is an intentional option and it is done with the goal to improve the understandability of the text. Often the word family is also used when refereing the set of its elements.

The option for the use of families is also justified by the fact that there are several results and formulations using the concept of cartesian product.

Definition 1.1.1 *A choice function in a family $\mathcal{F}$ is a function with domain $\mathcal{F}$ such that the image of each A in $\mathcal{F}$ is an element of A.*

Definition 1.1.2 (Axiom of Choice) *Every family of non-empty sets has a choice function.*

The Axiom of Choice is some times stated in te following way: *The cartesian product of non-empty sets is non-empty.*

Definition 1.1.3 (Multiplicative Axiom) *For every set $\mathcal{C}$ which the elements are disjoint and non-empty, there is a set A such that, for every $C \in \mathcal{C}$, $A \cap C$ has one and only one element.*

These two axioms are no more than two versions of the same axiom. From now on, they are both called Axiom of Choice (**AC**).

Definition 1.1.4 (Well-Ordering Principle) *Every set can be well-ordered.*

The Well-Ordering Principle is equivalent to the Axiom of Choice [31, p.10].

Definition 1.1.5 (Axiom of Multiple Choice – MC) For every family $\mathcal{F}$ of non-empty sets, there is a function f such that, for all $X \in \mathcal{F}$, $\emptyset \neq f(X) \subseteq X$ and $f(X)$ is finite.

Theorem 1.1.6 ([31, p.133]) *In* ZF*, the Axiom of Choice is equivalent to the Axiom of Multiple Choice.*

This theorem is not valid in ZFA (Second Fraenkel Model – $\mathcal{N}2$ in [29]). For details, see [34] or [31, p.135].

Proposition 1.1.7 ([34]) *The Axiom of Multiple Choice holds if and only if every set can be written as the well-ordered union of finite sets.*

Now, some weak forms of the Axiom of Choice will be introduced.

Definition 1.1.8 (Axiom of Countable Choice – CC) Every countable family of non-empty sets has a choice function.

It is not known if in ZF the Axiom of Countable Choice is equivalent to the Axiom of Multiple Countable Choice – **CMC**(defined similarly to the Axiom of Multiple Choice).

Definitions 1.1.9

(a) $\mathbf{CC}(\mathbb{R})$ states that the Axiom of Countable Choice holds for families of sets of real numbers.

(b) $\mathbf{AC}(\mathbb{R})$ states that the Axiom of Choice holds for families of sets of real numbers.

(c) $\mathbf{CC}(\alpha)$ states that the Axiom of Countable Choice holds for families of sets with cardinality at most equal α.

(d) $\mathbf{CC}(\text{fin})$ states that the Axiom of Countable Choice holds for families of finite sets.

We point out that $\mathbf{CC}(\mathbb{R})$ and $\mathbf{CC}(2^{\aleph_0})$ are not equivalent.

Forms of the kind $\mathbf{AC}(\alpha)$ or $\mathbf{MC}(\alpha)$ are also used with a meaning similar to the $\mathbf{CC}(\alpha)$.

Proposition 1.1.10 ([13, p.76], [23]) *The following conditions are equivalent to* **CC** *(respectively* $\mathbf{CC}(\mathbb{R})$*):*

(i) *every countable family of non-empty sets (resp. subsets of* $\mathbb{R}$*) has an infinite subfamily with a choice function;*

(ii) *for every countable family* $(X_n)_{n\in\mathbb{N}}$ *of non-empty sets (resp. subsets of* $\mathbb{R}$*), there is a sequence which intersects* X_n *for an infinite number of values of* n*.*

Definition 1.1.11 ($\mathbf{MC}_\omega$) For every family $\mathcal{F}$ of non-empty sets, there is a function f such that, for all $X \in \mathcal{F}$, $\emptyset \neq f(X) \subseteq X$ and $f(X)$ is countable.

Definitions 1.1.12

(a) $\mathbf{MC}_\omega(\mathbb{R})$ states that $\mathbf{MC}_\omega$ holds for families of sets of real numbers.

(b) $\mathbf{CMC}_\omega(\mathbb{R})$ states that $\mathbf{MC}_\omega(\mathbb{R})$ holds for countable families.

We consider also a form of "multiple" choice which is denoted by $\mathbf{CMC}_{WO}(\mathbb{R})$ which is $\mathbf{CMC}_\omega(\mathbb{R})$ replacing the choice of a countable set by a well-ordered set.

Proposition 1.1.13

(a) ([26]) *Every set can be written as the well-ordered union of countable sets if and only if* $\mathbf{MC}_\omega$ *holds.*

(b) ([17]) $\mathbb{R}$ *is the well-ordered union of countable sets if and only if* $\mathbf{MC}_\omega(\mathbb{R})$ *holds.*

Lemma 1.1.14

(a) $\mathbf{AC} \iff \mathbf{MC}_\omega + \mathbf{AC}(\aleph_0)$.

(b) $\mathbf{CC} \iff \mathbf{CMC} + \mathbf{CC}(\text{fin})$.

Other equivalences of the same type can be easily deduced.

In the absence of the Axiom of Choice there are several (non-equivalent) definitions of finite set. In this text we work only with two of them. A more general study of this subject can be found in [36, p.28] or [29, Note 94].

Definitions 1.1.15

(a) A set is finite if it is empty or equipollent to a natural number; otherwise is infinite.

(b) A set X is Dedekind-finite if no proper subset of X is equipollent to X; otherwise is Dedekind-infinite.

In the Cohen Basic Model, there is a subset of $\mathbb{R}$ which is infinite and Dedekind-finite.

Proposition 1.1.16 *A set* X *is Dedekind-infinite if and only if it has a countable subset, i.e. there is an injective function from* $\mathbb{N}$ *to* X.

Theorem 1.1.17 *Every infinite set is Dedekind-infinite if and only if the Axiom of Countable Choice holds for families of Dedekind-finite sets.*

Proposition 1.1.18 *The following conditions are equivalent:*

(i) *every infinite subset of* $\mathbb{R}$ *is Dedekind-infinite;*

(ii) *the Axiom of Choice holds for families of de Dedekind-finite subsets of* $\mathbb{R}$*;*

(iii) *the Axiom of Countable Choice holds for families of de Dedekind-finite subsets of* $\mathbb{R}$*;*

Definition 1.1.19 (Countable Union Condition – CUC) The countable union of countable sets is countable.

The condition "The countable union of finite sets is countable" is denoted by **CUC**(fin).

Lemma 1.1.20

(a) $\mathbf{CC}(2^{\aleph_0}) \implies \mathbf{CUC} \implies \mathbf{CC}(\aleph_0)$.

(b) $\mathbf{CUC}(\text{fin}) \iff \mathbf{CC}(\text{fin})$.

The implications of (a) are not reversible, once they are not true in Cohen's Basic Model ($\mathcal{M}1$ in [29]) and in Felgner Model I($\mathcal{M}20$ in [29]), respectively.

Definition 1.1.21 (Ultrafilter Theorem – UFT) Every filter over a set can extended to an ultrafilter.

This theorem is equivalent to the *Boolean Prime Ideal Theorem* , that is: every non-trivial ($0 \neq 1$) Boolean Algebra has a prime ideal. For details on this subject, see [31, 2.3].

We consider now some variations of the Ultrafilter Theorem.

Definitions 1.1.22

(a) ([22]) **CUF** states that the Ultrafilter Theorem holds for filters with a countable base.

(b) **CUF(**$\mathbb{R}$**)** states that the Ultrafilter Theorem holds for filters in $\mathbb{R}$ with a countable base.

Proposition 1.1.23 *If the Axiom of Countable Choice holds and* $\mathbb{N}$ *has a free ultrafilter, then* **CUF** *holds.*

Corollary 1.1.24 *The Ultrafilter Theorem is not equivalent to* **CUF**.

In Pincus Model IX ($\mathcal{M}47(n, M)$ in [29]) the Ultrafilter Theorem does not hold, but the Axiom of Countable Choice holds and there is a free ultrafilter in $\mathbb{N}$, and then **CUF** holds too.

Corollary 1.1.25 *If the Axiom of Choice holds for families of subsets of* $\mathbb{R}$, *then* **CUF**$(\mathbb{R})$ *holds.*

Theorem 1.1.26 $\mathbb{R}$ *has a free ultrafilter if and only if* $\mathbb{N}$ *has a free ultrafilter.*

In fact, there are several models where $\mathbb{N}$ has no free ultrafilters, for instance in Feferman's Model – $\mathcal{M}2$ in [29].

1.2 Topology

(...)

Definition 1.2.1 A topological space X is *(countably) compact* if every (countable) open cover of X has a finite subcover.

From this point we will use often the first separation axioms: T_0, T_1 and T_2 or Hausdorff. Their definitions, as well as their best known characterizations, are still valid in our axiomatic.

Proposition 1.2.2

(a) *Every closed subspace of a compact space is compact.*

(b) *Every compact subspace of a Hausdorff space is closed.*

Theorem 1.2.3 (Heine-Borel Theorem) *A subspace of* $\mathbb{R}$ *is compact if and only if it is closed and bounded.*

Definition 1.2.4 A topological space X is *sequentially compact* if every sequence in X has a convergent subsequence.

Definition 1.2.5 A topological space X is a *Lindelöf* space if every open cover of X has a countable subcover.

Definition 1.2.6 A topological space is *separable* if it has a countable dense subset.

Definitions 1.2.7

(a) A topological space X is *first countable* if every point of X has a countable neighborhood base[1].

(b) A topological space is *second countable* if it has a countable base.

In this text, we say frequently that a space has a countable space in stead of saying that it is second countable.

Proposition 1.2.8 *Every (countably) compact first countable space is sequentially compact.*

Lemma 1.2.9 *Let $(X, \mathcal{T})$ be a topological space.*

(a) *If $(X, \mathcal{T})$ is second countable, then $|\mathcal{T}| \leq |\mathbb{R}| = 2^{\aleph_0}$.*

(b) *If $(X, \mathcal{T})$ is a second countable T_0-space, then $|X| \leq |\mathbb{R}| = 2^{\aleph_0}$.*

Definitions 1.2.10 Let A be a subspace of the topological space X. The *sequential closure* of A in X is the set:

$$\sigma_X(A) := \{x \in X : (\exists (x_n) \in A^{\mathbb{N}})[(x_n) \text{ converges to } x]\}.$$

A is *sequentially closed* in X if $\sigma_X(A) = A$.

With the some kind of notation, the *(usual) Kuratowski closure* of A in X is denoted by $k_X(A)$.

Definitions 1.2.11 A topological space X is:

(a) *sequential* if, for all $A \subseteq X$, $\sigma_X(A) = A$ if and only if $k_X(A) = A$;

(b) a *Fréchet-Urysohn* space if, for all $A \subseteq X$, $k_X(A) = \sigma_X(A)$.

Note that a topological space X is sequential if and only if, for all $A \subseteq X$, $k_X(A) = \hat{\sigma}_X(A)$ with $\hat{\sigma}_X(A) := \bigcap\{B : A \subseteq B \subseteq X \text{ and } \sigma_X(B) = B\}$. This is the smallest idempotent closure operator that is bigger than σ. Immediately we have that, for all $A \subseteq X$, $\sigma_X(A) \subseteq \hat{\sigma}_X(A) \subseteq k_X(A)$.

[1] In the Portuguese version neighborhood base is often denoted by SFV.

Proposition 1.2.12 *A topological space X is a Fréchet-Urysohn space if and only if it is sequential and the sequential closure σ_X is idempotent.*

This Proposition just say that $k_X = \sigma_X$ if and only if $k_X = \hat{\sigma}_X$ and $\hat{\sigma}_X = \sigma_X$.

In this text, a pseudometric space is just a metric space in which the distance between two distinct points can be zero.

Definition 1.2.13 A (pseudo)metric space is *complete* if every Cauchy sequence converges.

Proposition 1.2.14

(a) *Every complete subspace of a metric space is sequentially closed.*

(b) *Every sequentially closed subspace of a complete (pseudo)metric space is complete.*

Note that (a) is not valid for pseudometric spaces.

Corollary 1.2.15 *Let X be a complete metric space. A subspace A of X is complete if and only if $\sigma_X(A) = A$.*

Corollary 1.2.16 *Every closed subspace of a complete (pseudo)metric space is complete.*

Corollary 1.2.17 *For a complete metric space X are equivalent:*

(i) *X is a sequential space;*

(ii) *every space $A \subseteq X$ is complete if and only if it is closed in X.*

Proposition 1.2.18 *Every sequentially compact (pseudo)metric space is complete.*

Corollary 1.2.19 *Every sequentially compact subspace of a metric space is sequentially closed.*

The result of this corollary is still true for a Hausdorff space in place of a metric space, what is an adaptation of Proposition 1.2.2.

It is also true that *every sequentially closed subspace of a sequentially compact space is sequentially compact.*

Chapter 2

The real numbers

2.1 $\mathbb{R}$ is a Fréchet-Urysohn space

Theorem 2.1.1 ([12, 24, 18]) *The following conditions are equivalent to* $\mathbf{CC}(\mathbb{R})$*:*

(i) $\mathbb{R}$ *is a Fréchet-Urysohn space;*

(ii) *the sequential closure is idempotent in* $\mathbb{R}$*.*

Corollary 2.1.2 ([12, p.128], [24]) *The following conditions are equivalent to* $\mathbf{CC}(\mathbb{R})$*:*

(i) *every subspace of* $\mathbb{R}$ *is separable;*

(ii) *every unbounded subset of* $\mathbb{R}$ *contains an unbounded countable set.*

Proposition 2.1.3 *The following conditions are equivalent:*

(i) *every infinite subset of* $\mathbb{R}$ *is Dedekind-infinite;*

(ii) *every set dense in* $\mathbb{R}$ *is Dedekind-infinite;*

(iii) *the Axiom of Choice holds for families of Dedekind-finite sets dense in* $\mathbb{R}$*;*

(iv) *the Axiom of Countable Choice holds for families of Dedekind-finite sets dense in* $\mathbb{R}$*.*

Proposition 2.1.4 *The following conditions are equivalent to* $\mathbf{CC}(\mathbb{R})$*:*

(i) *the Axiom of Countable Choice holds for families of sets dense in* $\mathbb{R}$*;*

(ii) *every subspace of* $\mathbb{R}$ *is separable;*

(iii) *every space dense in* $\mathbb{R}$ *is separable.*

Theorem 2.1.5 ([24, 1, 29]) *the discrete space* $\mathbb{N}$ *is Lindelöf if and only if the Axiom of Countable Choice holds for families of sets of real numbers.*

Corollary 2.1.6 ([24]) *The following conditions are equivalent to* $\mathbf{CC}(\mathbb{R})$*:*

(i) *every second countable space is a Lindelöf space;*

(ii) *every subspace of* $\mathbb{R}$ *is a Lindelöf space;*

(iii) $\mathbb{R}$ *is a Lindelöf space.*

Corollary 2.1.7 *Every separable subspace of* $\mathbb{R}$ *is Lindelöf if and only if* $\mathbf{CC}(\mathbb{R})$ *holds.*

Theorem 2.1.8 ([17])

(a) *Every unbounded Lindelöf subspace of* $\mathbb{R}$ *contains an unbounded sequence.*

(b) *For every Lindelöf subspace* A *of* $\mathbb{R}$*, if* $x \in k_{\mathbb{R}}(A)$ *then* $x \in \sigma_{\mathbb{R}}(A)$*.*

Theorem 2.1.9 ([21, 17]) *The following conditions are equivalent to* $\mathbf{CC}(\mathbb{R})$*:*

(i) *there is a Lindelöf non-compact subspace of* $\mathbb{R}$*;*

(ii) *there is a Lindelöf non-closed subspace of* $\mathbb{R}$*;*

(iii) *there is a Lindelöf unbounded subspace of* $\mathbb{R}$*.*

Lemma 2.1.10 ([35]) *The family of all non-empty closed subsets of* $\mathbb{R}$ *has a choice function.*

Theorem 2.1.11

(a) *Every closed subspace of* $\mathbb{R}$ *is separable.*

(b) *Every compact subspace of* $\mathbb{R}$ *is separable.*

(c) *Every Lindelöf subspace of* $\mathbb{R}$ *is separable.*

2.2 $\mathbb{R}$ is a sequential space

Theorem 2.2.1 ([17]) *If* $\mathbb{R}$ *is sequential, then every infinite subset of* $\mathbb{R}$ *is Dedekind-infinite.*

Proposition 2.2.2 $\mathbb{R}$ *is sequential if and only if every complete subspace of* $\mathbb{R}$ *is closed.*

Theorem 2.2.3 ([17]) *Every complete subspace of* $\mathbb{R}$ *is closed if and only if* $\mathbb{R}$ *the complete dense subspace of* $\mathbb{R}$*.*

In other words, if there is $A \subseteq \mathbb{R}$ complete but not closed, then there is $B \neq \mathbb{R}$ complete and dense in $\mathbb{R}$.

Lemma 2.2.4 *A bounded subspace of $\mathbb{R}$ is sequentially compact if and only if it is complete.*

Theorem 2.2.5 ([17]) *The following conditions are equivalent:*

(i) *$A \subseteq \mathbb{R}$ is complete if and only if it is closed;*

(ii) *if $A \subseteq \mathbb{R}$ is sequentially compact, then it is closed;*

(iii) *if $A \subseteq \mathbb{R}$ is complete, then it is separable;*

(iv) *if $A \subseteq \mathbb{R}$ is sequentially compact, then it is separable.*

Proposition 2.2.6 ([17]) *The following conditions are equivalent:*

(i) *a subspace of $\mathbb{R}$ is sequentially compact if and only if it is compact;*

(ii) *every sequentially compact subspace of $\mathbb{R}$ is closed;*

(iii) *every sequentially compact subspace of $\mathbb{R}$ is bounded.*

Lemma 2.2.7 *Every sequentially compact Lindeöf subspace of $\mathbb{R}$ is compact.*

Proposition 2.2.8 ([17]) *Every sequentially compact subspace of $\mathbb{R}$ is compact if and only if every sequentially compact subspace of $\mathbb{R}$ is Lindelöf.*

Lemma 2.2.9 ([17]) *$\mathbb{R}$ is a sequential space if and only if the Axiom of (Countable) Choice holds for families of sequentially closed (=complete) subspaces of $\mathbb{R}$.*

Theorem 2.2.10 ([17]) *If $\mathbf{CMC}_{WO}(\mathbb{R})$ holds, then $\mathbb{R}$ is a sequential space.*

Corollary 2.2.11 *If $\mathbb{R}$ is the countable union of countable sets, then $\mathbb{R}$ is a sequential space.*

Corollary 2.2.12 ([17]) *The condition "$\mathbb{R}$ is a sequential space" does not imply $\mathbf{CC}(\mathbb{R})$.*

In [6] A. Church had proved that $\mathbf{CC}(\mathbb{R})$ implies that the first uncountable ordinal is not the limite of a sequence of countable ordinals, which implies that $\mathbb{R}$ is not the countable union of countable sets (see [31, p.148]). In Feferman/Levy Model [11], $\mathbb{R}$ is the countable union of countable sets, and then $\mathbf{CC}(\mathbb{R})$ does not hold, but by 2.2.11 $\mathbb{R}$ is sequential.

Using the same kind of argument, one can deduce the following corollary. This simplifies the proof of [27].

Corollary 2.2.13 *If $\mathbb{R}$ is the countable union of countable sets, then the Axiom Countable Choice does not hold for families of countable subsets of $\mathbb{R}$.*

Chapter 3

Countable Properties

3.1 Metric spaces

(...)

Lemma 3.1.1 *Every pseudometric separable space has a countable base.*

Theorem 3.1.2 *If the Axiom of Countable Choice holds, then, for a pseudometric space X, the following conditions are equivalent:*

(i) *X is a Lindelöf space;*

(ii) *X has a countable base;*

(iii) *X is separable.*

Proposition 3.1.3 *The following conditions are equivalent to* $\mathbf{CC}(\mathbb{R})$*:*

(i) *every (pseudo)metric space with a countable base is a Lindelöf space;*

(ii) *every (pseudo)metric space separable is a Lindelöf space.*

Proposition 3.1.4 *the following conditions are equivalent to* $\mathbf{CC}(\mathbb{R})$*:*

(i) *every T_0-space with a countable base is separable;*

(ii) *a metric space has a countable base if and only if it is separable;*

(iii) *every subspace of a separable metric space is separable.*

Proposition 3.1.5 ([2]) *The following conditions are equivalent to* $\mathbf{CC}$*:*

(i) *every topological (pseudometric) space with a countable base is separable;*

(ii) *every Lindelöf pseudometric space is separable;*

(iii) *every Lindelöf topological (pseudometric) space with a countable base is separable.*

Proposition 3.1.6 ([2]) *The Axiom of Countable Choice holds if and only if every subspace of separable pseudometric space is separable.*

Together with the questions we have just studied there are three that are interesting to study in ZF: a) every Lindelöf metric space is separable; (b) every Lindelöf metric space has a countable base; (c) every subspace of a Lindelöf metric space is a Lindelöf space.

The final results of this section show that these three conditions are not provable in ZF.

Proposition 3.1.7 ([15]) *If every Lindelöf metric space has a countable base, then the Axiom Countable Choice holds for families of finite sets (*$\mathbf{CC}(\mathrm{fin})$*).*

Corollary 3.1.8 *If every Lindelöf metric space is separable, then* $\mathbf{CC}(\mathrm{fin})$ *holds.*

Corollary 3.1.9 *If every subspace of a Lindelöf metric space is a Lindelöf space, then* $\mathbf{CC}(\mathrm{fin})$ *holds.*

It is also true that, if every subspace of a Lindelöf metric space is a Lindelöf space, then $\mathbf{CC}(\mathbb{R})$ holds (see 2.1.6 or 2.1.9).

3.2 Countable base

Next, it will be studied a characterization of the second countable spaces , which is not provable in ZF. We start with the theorem that gives us that characterization in ZFC, and after that it is investigated the degree of choice needed for the proof.

Theorem 3.2.1 (ZFC) *Every base of a second countable space contains a countable base.*

Theorem 3.2.2 ([16]) *The following conditions are equivalent to* $\mathbf{CC}(\mathbb{R})$*:*

(i) *every base of a second countable space contains a countable base;*

(ii) *every base for the open sets of the topological space* $\mathbb{R}$ *contains a countable base.*

Definition 3.2.3 A topological space is *super second countable* (SSC)[1] if every base contains a countable base.

Corollary 3.2.4 *The following conditions are equivalent to* $\mathbf{CC}(\mathbb{R})$*:*

(i) $\mathbb{R}$ *is SSC;*

(ii) *every separable (pseudo)metric space is SSC.*

[1]SBN in the Portuguese version.

The condition *every SSC topological (or pseudometric) space is separable* is equivalent to **CC**.

Theorem 3.2.5 *Every SSC subspace of* $\mathbb{R}$ *is separable.*

The proof that *every metric space with a countable base is separable* implies $\mathbf{CC}(\mathbb{R})$ (3.1.4) relies on the fact that this is not true for subspaces of $\mathbb{R}$. As consequence of that, the proof cannot be adapted for SSC spaces. It is interesting to ask if there is any non-separable SSC metric space.

Theorem 3.2.6 *Every Lindelöf subspace of* $\mathbb{R}$ *is SSC if and only if* $\mathbf{CC}(\mathbb{R})$ *holds.*

Corollary 3.2.7 *If every Lindelöf metric space is SSC, then* $\mathbf{CC}(\mathbb{R})$ *holds.*

One easily see that the condition *"Every SSC space is Lindelöf"* is equivalent to $\mathbf{CC}(\mathbb{R})$, once $\mathbb{N}$ is SSC.

A question still open is to know if or when the property of being SSC is hereditary.

3.3 Products of second countable and separable spaces

In ZFC the countable product of second countable spaces is second countable, and the countable product of Hausdorff separable spaces is separable. These two conditions were considered by K. Keremedis [33] that proved that any of them imply the Axiom of Countable Choice for families of countable sets ($\mathbf{CC}(\aleph_0)$). In this section we will improve his results and at same time we consider generalizations and restrictions of these conditions.

Lemma 3.3.1 *Let* $(X_n)_n$ *be a countable family of countable sets.*

The union $\bigcup_n X_n$ *is countable if and only there is a countable family of injective functions* $(f_n : X_n \longrightarrow \mathbb{N})_n$.

Proposition 3.3.2 ([16]) *If the countable product of second countable spaces is second countable, then* **CUC** *does hold.*

Proposition 3.3.3 *If the countable product of separable Hausdorff spaces is separable, then* **CUC** *does hold.*

Corollary 3.3.4 *If* $\prod_{i\in\mathbb{R}} X_i$ *is separable for any family* $(X_i)_{i\in\mathbb{R}}$ *of separable Hausdorff spaces, then for every family* $(Y_i)_{i\in\mathbb{R}}$ *of countable sets there is a family* $(f_i : Y_i \longrightarrow \mathbb{N})_{i\in\mathbb{R}}$ *of injective functions.*

Proposition 3.3.5 *If* $\mathbf{CC}(2^{\aleph_0})$ *holds, then the countable product of second countable spaces is second countable.*

Corollary 3.3.6 *The countable product of spaces with finite topologies is second countable if and only if the Axiom of Countable Choice holds for families of finite sets (*$\mathbf{CC}(\mathrm{fin})$*).*

Theorem 3.3.7 *It is separable the product* $\prod_{i\in\mathbb{R}} X_i$ *of any family* $(X_i)_{i\in\mathbb{R}}$ *of separable spaces if and only if the Axiom Choice holds for families with index at most* $2^{\aleph_0}$.

Corollary 3.3.8 *The countable product of separable spaces is separable if and only if the Axiom of Countable Choice does hold.*

Corollary 3.3.9

(a) *The product* $\prod_{i\in\mathbb{R}} X_i$ *of any family* $(X_i)_{i\in\mathbb{R}}$ *of spaces with finite topologies is separable if and only if the Axiom of Choice holds for families with index at most* $2^{\aleph_0}$.

(b) *The countable product of spaces with finite topologies is separable if and only if* $\mathbf{CC}$ *does hold.*

(c) *It is separable the product* $\prod_{i\in\mathbb{R}} X_i$ *of any family* $(X_i)_{i\in\mathbb{R}}$ *of finite spaces if and only if* $\mathbf{AC}(\mathrm{fin})$ *holds for families with index at most* $2^{\aleph_0}$.

(d) *The countable product of finite spaces is separable if and only if* $\mathbf{CC}(\mathrm{fin})$ *does hold.*

3.4 Products of Lindelöf spaces

Theorem 3.4.1 ([21]) *There is a Lindelöf non-compact* T_1*-space if and only if* $\mathbf{CC}(\mathbb{R})$ *does hold.*

Theorem 3.4.2 ([38, 37, 20]) *The following conditions are equivalent:*

(i) *the product of Hausdorff compact spaces is compact;*

(ii) 2^I *is compact for any* I*, where* 2 *is the discrete space with* 2 *points;*

(iii) *the Ultrafilter Theorem.*

Lemma 3.4.3 (e.g., [43, 103.6], [21]) $\mathbb{N}^{\mathbb{R}}$ *is not a Lindelöf space.*

Theorem 3.4.4 ([21]) *The product of Hausdorff Lindelöf spaces is Lindelöf if and only if the Ultrafilter Theorem holds but* $\mathbf{CC}(\mathbb{R})$ *does not.*

There are in fact models where UFT holds and $\mathbf{CC}(\mathbb{R})$ does not hold (Cohen Basic Model– $\mathcal{M}1$ in [29]).

Proposition 3.4.5 *If 2^I is a Lindelöf space for any I, then either* **UFT** *holds or* $\mathbf{CC}(\mathbb{R})$ *holds.*

In Truss' Model I ($\mathcal{M}12(\aleph)$ in [29]), there is a set I such that 2^I is not Lindelöf, because neither **UFT** nor $\mathbf{CC}(\mathbb{R})$ are propositions of this model.

For more information about products of Lindelöf spaces in ZF, you should see [21].

Chapter 4

Spaces defined by limits

4.1 Spaces defined by sequences

In this section, we investigate in what conditions each of these classes is contained in the class of the Fréchet-Urysohn spaces or in the class of sequential spaces:

(a) first countable spaces,

(b) (pseudo)metric spaces,

(c) second countable spaces,

(d) second countable T_0-spaces,

(e) second countable metric spaces,

(f) subspaces of $\mathbb{R}$.

As it was done for real numbers, it will be also studied the class of spaces with idempotent sequential closure, which has a clear connection with the other two classes mentioned.

Theorem 4.1.1 *Every metric space is a sequential space if and only if the Axiom of Countable Choice holds.*

Corollary 4.1.2 *The following conditions are equivalent to* **CC***:*

(i) *every complete metric space is sequential;*

(ii) *every complete subspace of a metric space is closed.*

Corollary 4.1.3 *The following conditions are equivalent to* **CC***:*

(i) *every (pseudo)metric space is a Fréchet-Urysohn space;*

(ii) *every first countable space is Fréchet-Urysohn;*

(iii) *every first countable space is sequential.*

Theorem 4.1.4 *Every second countable space is sequential if and only if the Axiom of Countable Choice holds.*

Corollary 4.1.5 *Every second countable space is Fréchet-Urysohn if and only if the Axiom of Countable Choice holds.*

Theorem 4.1.6 *The sequential closure is idempotent for all metric spaces if and only if the Axiom of Countable Choice holds.*

Corollary 4.1.7 *The sequential closure is idempotent for all first countable spaces if and only if the Axiom of Countable Choice does hold.*

Theorem 4.1.8 *The sequential closure is idempotent for all second countable spaces if and only if the Axiom of Countable Choice does hold.*

Proposition 4.1.9 *The following conditions are equivalent to* $\mathbf{CC}(\mathbb{R})$*:*

(i) *every second countable* T_0*-space is Fréchet-Urysohn;*

(ii) *the sequential closure is idempotent for every second countable* T_0*-space;*

(iii) *every second countable metric space is Fréchet-Urysohn;*

(iv) *the sequential closure is idempotent for every second countable metric space;*

(v) *every subspace of* $\mathbb{R}$ *is a Fréchet-Urysohn space;*

(vi) *the sequential closure is idempotent for every subspace of* $\mathbb{R}$.

Theorem 4.1.10 *Every subspace of* $\mathbb{R}$ *is sequential if and only if* $\mathbf{CC}(\mathbb{R})$ *does hold.*

Corollary 4.1.11 *The following conditions are equivalent to* $\mathbf{CC}(\mathbb{R})$*:*

(i) *every second countable* T_0*-space is sequential;*

(ii) *every second countable metric space is sequential.*

4.2 Compact spaces

In the absence of the Axiom of Choice several usual characterizations and results about compact spaces do not remain valid. This is in fact one of the most studied subjects in *choicefree topology* (e.g. [20, 8]).

We center our attention in results concerning sequentially compact spaces because they are related with the questions studied in the previous section.

(...)

Lemma 4.2.1 *If the Axiom of Countable Choice holds, every sequentially compact space is countably compact.*

Theorem 4.2.2 *If the Axiom of Countable Choice holds, then a second countable space is compact if and only if it is sequentially compact.*

Theorem 4.2.3 *If the Axiom of Countable Choice holds, then a pseudometric space is compact if and only if it is sequentially compact.*

Proposition 4.2.4 *Every sequentially compact space is countably compact if and only if the Axiom of Countable Choice does hold.*

Corollary 4.2.5 ([2]) *The following conditions are equivalente to* **CC***:*

(i) *a first countable space is sequentially compact if and only if it is countably compact;*

(ii) *a second countable space is sequentially compact if and only if it is (countably) compact;*

(iii) *a pseudometric space is sequentially compact if and only if it is (countably) compact;*

(iv) *every sequentially compact pseudometric space is bounded.*

Theorem 4.2.6 *If every sequentially compact metric space is bounded, then every infinite set is Dedekind-infinite.*

Corollary 4.2.7 *Each of the following conditions implies that every infinite set is Dedekind-infinite:*

(i) *every sequentially compact metric space is (countably) compact;*

(ii) *every sequentially compact metric space is a Lindelöf space;*

(iii) *every sequentially compact subspace of a metric space is closed.*

Proposition 4.2.8 *Every second countable sequentially compact space is a Lindelöf space if and only if the Axiom of Countable Choice holds for families of subsets of* $\mathbb{R}$*.*

Corollary 4.2.9 *If every sequentially compact pseudometric space is Lindelöf, then* $\mathbf{CC}(\mathbb{R})$ *holds and every infinite is Dedekind-infinite.*

Proposition 4.2.10 ([33]) *If every countably compact metric space is compact, then every infinite set is the countable union of disjoint non-empty sets. vazios.*

In Model Truss III ($\mathcal{M}37$ in [29]) there is an infinite set which it is not the disjoint union of two infinite sets (e.g. [31, p.95], [29, Forma 64]), and then it is not the countable union of disjoint non-empty sets. This means that in this model there is a countably compact metric space which is not compact.

4.3 Spaces defined by ultrafilters

In this section, our first goal is to show that the following Theorem of ZFC is equivalente to the Ultrafilter Theorem (**UFT**).

Theorem 4.3.1 (ZFC) *The point $x \in X$ is in the closure of A in X if and only if there is an ultrafilter $\mathcal{U}$ in X such that $\mathcal{U}$ converges to x and $A \in \mathcal{U}$.*

As it was done for sequences, one consider the closure operator u defined by limits of ultrafilters and its idempotent hull $\hat{u}$.

Definitions 4.3.2 Let A be a subspace of the topological space X.

(a) $u_X(A) := \{x \in X : (\exists \mathcal{U} \text{ ultrafilter in } X)[\mathcal{U} \text{ converges to } x \text{ and } A \in \mathcal{U}]\}$.

(b) $\hat{u}_X(A) := \bigcap\{B : A \subseteq B \text{ and } u_X(B) = B\}$.

In parallel with section 4.1, and using this time the closure u, one can study when each of the equalities $u = k$[1] e $\hat{u} = k$[2] hold in these classes:

(a) topological spaces,

(b) Hausdorff spaces,

(c) first countable spaces,

(d) (pseudo)metric spaces,

(e) second countable spaces,

(f) second countable T_0-spaces,

(g) second countable metric spaces,

(h) subspaces of $\mathbb{R}$,

(i) $\{\mathbb{R}\}$.

The idempotency of u ($u = \hat{u}$) cannot be studied together with the other two properties. That case will be analyzed in the end of this section.

Theorem 4.3.3 *The closure operators $\hat{u}$ and k coincide in the class of the Hausdorff spaces if and only if the Ultrafilter Theorem(***UFT***) holds.*

[1] X is Fréchet-Urysohn if and only if $k_X = \sigma_X$.
[2] X is sequential if and only if $k_X = \hat{\sigma}_X$.

Corollary 4.3.4 *The following conditions are equivalente to* **UFT***:*

(i) $u = k$ *in the class of the topological spaces;*

(ii) $\hat{u} = k$ *in the class of the topological spaces;*

(iii) $u = k$ *in the class of the Hausdorff spaces.*

The condition (i) of this corollary is the Proposition 4.3.1.

Proposition 4.3.5 *The following conditions are equivalente to* **CUF***:*

(i) $u = k$ *in the class of the first countable spaces;*

(ii) $\hat{u} = k$ *in the class of the first countable spaces;*

(iii) $u = k$ *in the class of the metric spaces;*

(iv) $\hat{u} = k$ *in the class of the metric spaces.*

Proposition 4.3.6 *The following conditions are equivalent to* **CUF***:*

(i) $u = k$ *in the class of the second countable spaces;*

(ii) $\hat{u} = k$ *in the class of the second countable spaces.*

Theorem 4.3.7 *The closure operators* u *and* k *coincide in* $\mathbb{R}$ *if and only if* **CUF**$(\mathbb{R})$ *does hold.*

Corollary 4.3.8 *The following conditions are equivalente to* **CUF**$(\mathbb{R})$*:*

(i) $u = k$ *in the class of the second countable* T_0*-spaces;*

(ii) $\hat{u} = k$ *in the class of the second countable* T_0*-spaces;*

(iii) $u = k$ *in the class of the second countable metric spaces;*

(iv) $\hat{u} = k$ *in the class of the second countable metric spaces;*

(v) $u = k$ *for subspaces of* $\mathbb{R}$*;*

(vi) $\hat{u} = k$ *for subspaces of* $\mathbb{R}$*.*

Lemma 4.3.9 *For every topological space* $(X, \mathcal{T})$ *such that* X *has no free ultrafilters and for all* $A \subseteq X$*:*

$$u_X(A) = \bigcup_{a \in A} k_X(\{a\}).$$

If $(X, \mathcal{T})$ *a* T_1*-space, then* $u_X(A) = A$*.*

There are in fact models of ZF with sets which have no free ultrafilters or, even more, with no free ultrafilters at all. A. Blass [3] built a model ($\mathcal{M}15$ in [29]) where every ultrafilters are fixed. For details, see Forms 63 and 206 of [29].

Proposition 4.3.10 *If $\mathbb{R}$ has no free ultrafilters, then $\hat{u}_{\mathbb{R}} \neq k_{\mathbb{R}}$.*

From this proposition, one concludes that $\hat{u}_{\mathbb{R}} = k_{\mathbb{R}}$ is not a theorem of ZF.

There are models of ZF where $\mathbb{R}$ has no free ultrafilters, but other sets have. The Feferman Model is an example of that, as it was already said (1.1.26).

Proposition 4.3.11 *If every ultrafilter in X is fixed, then u_X is idempotent.*

Corollary 4.3.12 *The following sentences are consistent with* ZF*.*

(a) *$\hat{u} = u$ in the class of the topological spaces and* **UFT** *does not hold.*

(b) *$\hat{u} = u$ in the class of the first countable spaces and* **CUF** *does not hold.*

(c) *$\hat{u}_{\mathbb{R}} = u_{\mathbb{R}}$ and* **CUF($\mathbb{R}$)** *does not hold.*

These three conditions are true in the Feferman/Blass Model ($\mathcal{M}15$ em [29]).

4.4 Hausdorff spaces

Theorem 4.4.1 *The Axiom of Countable Choice is equivalent to:*

(i) *a first countable space is Hausdorff if and only if each of its sequences has at most one limit.*

Theorem 4.4.2 *The Axiom of Countable Choice for families of subsets of $\mathbb{R}$ (**CC**($\mathbb{R}$)) is equivalent to:*

(i) *a second countable space is Hausdorff if and only if each of its sequences has at most one limit.*

Theorem 4.4.3 *The Ultrafilter Theorem (**UFT**) is equivalent to:*

(i) *a topological space is Hausdorff if and only if each of its ultrafilters has at most one limit.*

Corollary 4.4.4 *the following condition is equivalent to* **CUF***:*

(i) *a first countable space is Hausdorff if and only if each of its ultrafilters has at most one limit.*

Corollary 4.4.5 *The following condition is equivalent to* **CUF($\mathbb{R}$)***:*

(i) *a second countable space is Hausdorff if and only if each of its ultrafilters has at most one limit.*

Chapter 5

Complete metric spaces

The results presented in this chapter, namely in the first and second section, are motivated by the Corollary 4.1.2. This corollary says that *every complete metric space is a sequential space if and only if the Axiom of Countable Choice holds.* In other words, in the absence of the Axiom of Countable Choice there is a sequentially closed not closed subspace of a complete metric space. Such a space has two non-isometric completions, because the usual definition of completion says that Y is a completion of X if it is complete and X is dense in Y.

5.1 Completions

One of the most usual proves for the existence of a completion of a metric space (X, d) is done considering the set $\mathcal{H}$ of all of its Cauchy sequences. After that, one takes in $\mathcal{H}$ the equivalence relation defined in the following way: $(x_n)_n \sim (y_n)_n$ if $\lim d(x_n, y_n) = 0$. One defines also a metric $\tilde{d}$ in $\mathcal{H}/_\sim$ such that $\tilde{d}([(x_n)], [(y_n)]) := \lim d(x_n, y_n)$. It is clear that X is isometric to a dense subspace of $\mathcal{H}/_\sim$. In order to show that the metric space constructed in this way is complete, one needs the Axiom of Countable Choice.

This construction makes automatically the completion unique.

A peculiar case of the construction that we just described is the construction of the real numbers from the rational numbers by means of Cauchy sequences. In that case there is no problem with choice, since $\mathbb{Q}$ is a countable set (e.g. [39]).

One should remark that an iteration of this process does not give for free a complete space. In fact, M. Gitik [14] constructed a model where all the limit ordinals are the limit of a sequence of smaller ordinals. This means that constructions of this kind might be done indefinitely (see also [29, Forma 182]).

One can conclude that the uniqueness of the completion is a consequence from the equality $\hat{\sigma} = k$, and at least one construction of completion is a consequence of the equality $\sigma = \hat{\sigma}$. That is exactly the reason why in ZFC the completion exists and it is unique, since the metric spaces are Fréchet-Urysohn spaces, that is $\sigma = k$.

In this point, we introduce three definitions of completion. The completions should be considered to less than a isometry.

Definitions 5.1.1 Let X be a complete metric space and A one of its subspaces. One says that X is a:

(a) σ-completion of A if $\sigma_X(A) = X$;

(b) $\hat{\sigma}$-completion of A if $\hat{\sigma}_X(A) = X$;

(c) k-completion of A if $k_X(A) = X$.

The definition of k-completion is the usual.

A σ-completion exists only when the construction that we did before produces a complete space.

Since the sequentially closed subspaces of a complete metric space X are exactly its complete subspaces (Proposition 1.2.15), if $\hat{\sigma}_X(A) = X$, there is no complete space between A and X. One concludes that a $\hat{\sigma}$-completion is minimal.

Proposition 5.1.2 *Every metric space has a $\hat{\sigma}$-completion.*

Corollary 5.1.3 *Every metric space has a k-completion.*

Definition 5.1.4 Let A be a subspace of a topological space X. For any ordinal α, define:

$$\sigma_X^\alpha(A) := \begin{cases} \sigma_X(A) & \text{if } \alpha = 1 \\ \sigma_X(\sigma_X^\beta(A)) & \text{if } \alpha = \beta + 1 \\ \bigcup\{\sigma_X^\beta(A) : \beta < \alpha\} & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Proposition 5.1.5 *Let A be a subspace of a topological space X.*

$$\hat{\sigma}_X(A) = \sigma_X^\alpha(A), \textit{ for } \alpha := \min\{\beta : \sigma_X^{\beta+1}(A) = \sigma_X^\beta(A)\}.$$

Theorem 5.1.6 *The $\hat{\sigma}$-completion of a metric space is unique.*

Corollary 5.1.7 *Let A be a metric space and X its $\hat{\sigma}$-completion.*

If f is a non-expansive map[1] *from A to a complete metric space B, then there is a unique non-expansive function $\hat{f}$ from X to B such that its restriction to A is f.*

[1] $f : (A, d) \longrightarrow (B, d')$ is a non-expansive map if $d'(f(x), f(y)) \leq d(x, y)$.

In other words, the complete metric spaces are a reflexive subcategory of the category of metric spaces and non-expansive maps.

The result is still valid when the morphisms are the uniformly continuous functions.

Proposition 5.1.8 *Every metric space has a unique k-completion if and only if the Axiom of Countable Choice holds.*

Proposition 5.1.9 *Every metric space has a σ-completion if and only if the Axiom of Countable Choice holds. If the the σ-completion exists, then it is unique.*

5.2 f-complete spaces

As we have seen before, of absence of choice might be turned replacing sequences by filters.

That is, in fact, what happens in the definition of complete uniform space, since the uniform spaces are not sequential, even in ZFC. In ZF the metric spaces might be not sequential, and then it is natural to consider the spaces *complete for filters* or *f-complete*.

Definition 5.2.1 A metric space is *f-complete* if every *Cauchy filter*[2] converges.

Proposition 5.2.2 *A subspace of a f-complete space is f-complete if and only if it is closed.*

Definition 5.2.3 Let X be a f-complete space and $A \subseteq X$. One says that X is a f-completion of A if $k_X(A) = X$.

Theorem 5.2.4 *Every metric space has a unique f-completion.*

The completion of a uniform space might be constructed from a equivalence relation in the set of their Cauchy filters (for details see [30, p.155]).

An alternative proof of the existence of the completion from a uniform is done from the fact that *every uniform space is uniformly equivalent to a subspace of the uniform product of metrizable spaces* (e.g. [10, 8.2.3]). Although the Axiom of Multiple Choice is used in the proof of this result. If one consider the definition of uniform spaces made in terms of gauges of pseudometrics, this second approach turns out to be more appropriated.

Corollary 5.2.5 *The Axiom of Countable Choice is equivalent to:*

(i) *a metric space is complete if and only if it is f-complete.*

[2] $\mathcal{F}$ is a Cauchy filter if, for every n, there is $F \in \mathcal{F}$ such that $\mathsf{diam}(F) < \frac{1}{n}$.

Definition 5.2.6 A metric space X is *Cantor-complete* if, for every family $(F_n)_{n\in\mathbb{N}}$ of non-empty closed subsets of X such that $F_{n+1} \subseteq F_n$ and $\lim_n \mathsf{diam}(F_n) = 0$, $\bigcap_n F_n \neq \emptyset$.

Theorem 5.2.7 *For a metric space X, the following conditions are equivalent:*

(i) *X is f-complete;*

(ii) *every Cauchy filter in X with a countable base converges;*

(iii) *X is Cantor-complete.*

5.3 Products

The products of (pseudo)metric spaces might be formed in respect to their topological, uniform or metric structure. In the two first cases, these products are not, in general, metrizable and in the third they not always exist.

We will study the existence of the product metric, and also the stability of the complete (pseudo)metric spaces for that product.

Given a countable family of metric spaces $((X_n, d_n))_n$, its topological product is metrizable. The result holds in ZF, but the same it is not true for metrizable spaces, for which the proof uses a countable (multiple) choice.

Proposition 5.3.1 ([33]) *If the Axiom of Countable Multiple Choice (***CMC***) holds, then the countable product of metrizable spaces is metrizable.*

Proposition 5.3.2 ([33]) *If the countable product of metrizable spaces is metrizable, then* **CMC**$(\aleph_0)$ *holds.*

There are similar results to the first countable spaces ([33]).

In general, if the product of a family of metrizable non-empty spaces is metrizable, then all but a countable number of the factors are trivial spaces. Although, in ZF this condition is not always true because it is equivalent to the Axiom of Choice. The picture is completely different if one considers only the non-empty products.

Proposition 5.3.3 *Let $(X_i)_{i\in I}$ be a family of non-empty non-singular sets and, for each $i \in I$, $\mathcal{T}_i$ a metrizable topology in X_i. If* **CC**(fin) *holds, $\prod X_i \neq \emptyset$ and $\prod (X_i, \mathcal{T}_i)$ is metrizable, then $|I| \leq \aleph_0$.*

Theorem 5.3.4 *The existent products of complete pseudometric spaces are complete if and only if the Axiom of Choice holds.*

Corollary 5.3.5 ([40]) *The uniform product of complete uniform spaces is complete if and only if the Axiom of Choice holds.*

Chapter 6

First countable spaces

A topological space is first countable if there is a countable neighborhood base (or local base) at each of its points (Definition 1.2.7). In general, that is in the presence of the Axiom of Choice, this definition is clear and there is no room for two different interpretations. But what happens when the Axiom of Choice does not hold? The first consequence is that the definition does not give an algorithm to find, simultaneous, a countable neighborhood base at each point of a first countable space. The existence of a solution for this kind of problems is one of the reasons for the use of the Axiom of Choice.

(...)

6.1 Definitions

We start with the definition of three conditions, all of them equivalent to the First Axiom of Countability in ZFC. They will be denoted by A, B and C, being A the usual definition. Later some other conditions will be introduced and they will also be denoted in alphabetic order. In order to make the new definitions easier to understand and to compare, they are presented in symbolic language.

Definitions 6.1.1 Let X be a topological. One says that X satisfies:

A if $(\forall x \in X)\,(\exists \mathcal{B}(x))\ |\mathcal{B}(x)| \leq \aleph_0$ and $\mathcal{B}(x)$ is a local base at x;

B if $(\exists (\mathcal{B}(x))_{x \in X})\,(\forall x \in X)\ |\mathcal{B}(x)| \leq \aleph_0$ and $\mathcal{B}(x)$ is a local base at x;

C if $(\exists \{B(n,x) : n \in \mathbb{N},\, x \in X\})\,(\forall x)\ \{B(n,x) : n \in \mathbb{N}\}$ is a local base at x.

In the definitions of A, B and C, one can take only the open neighborhoods without changing the logic value of them. This fact is pointed out because it will be seen other situations where that is not the case.

Lemma 6.1.2

(a) *If a topological space satisfies* B*, then satisfies* A.

(b) *If a topological space satisfies* C*, then satisfies* B.

Proposition 6.1.3 *Every metric or second countable space satisfies* C*, and then also* B *and* A.

The first intuitive idea that one might have is that the equivalence between A and B needs the Axiom of Choice, because the implication A⇒B has some formal similarity to the Axiom of Choice. Although, it is possible to prove that A is equivalent to B from a choice principle weaker then **AC**.

Theorem 6.1.4 ([16]) *If* $\mathbf{MC}_\omega$ *holds, then a topological space satisfies* A *if and only if satisfies* B.

There are several models of ZF where **AC** does not hold, but $\mathbf{MC}_\omega$ does, for instance the Cohen/Pincus Model($\mathcal{M}1(<\omega_1>)$ in [29]).

Unfortunately, it is not known if the equivalence between A and B is provable in ZF.

Proposition 6.1.5 *If* $\mathbf{MC}(2^{\aleph_0})$ *holds, then a topological space satisfies* B *if and only if satisfies* C.

Lemma 6.1.6 *The following conditions are equivalent:*

(i) $\mathbf{AC}(\aleph_0)$*;*

(ii) *for every family of countable sets* $(X_i)_{i\in I}$*, there is a family of functions* $(f_i)_{i\in I}$ *such that* f_i *is a bijection between an ordinal* α_i *and* X_i*.*

Corollary 6.1.7 *If* $\mathbf{AC}(\aleph_0)$ *and* $\mathbf{AC}(\mathbb{R})$ *hold, then for every family of countable sets* $(X_i)_{i\in I}$*, there is a family of functions* $(f_i)_{i\in I}$ *such that* f_i *is an injection from* X_i *to* $\mathbb{N}$.

From this corollary and from Lemma 3.3.1, one has the next result.

Corollary 6.1.8 *If* $\mathbf{AC}(\aleph_0)$ *and* $\mathbf{AC}(\mathbb{R})$ *hold, then the countable union of countable sets is countable (***CUC***).*

Corollary 6.1.9 *If* $\mathbf{AC}(\aleph_0)$ *and* $\mathbf{AC}(\mathbb{R})$ *hold, then a topological space satisfies* B *if and only satisfies* C.

Proposition 6.1.10 *If a topological space satisfies* B *if and only if satisfies* C*, then* $\mathbf{MC}(\aleph_0)$ *holds.*

6.2 Characterizations

The motivation for the work presented in this section is the attempt to generalize the result of Theorem 3.2.2 for first countable spaces. That is, we try to see in what conditions we can take a countable local base from any local base in a first countable space.

Following what was done in the previous section, there are three ways to do it: one local and two global, in accordance with each of the definitions A, B e C.

Theorem 6.2.1 (ZFC) *Every neighborhood base at a point of a first countable space contains a countable neighborhood base.*

This is the usual version of the theorem. However, it is not necessary to consider a first countable space, it suffices to consider that a specific point has a countable neighborhood base. For that reason, perhaps it is more appropriate to consider a global version of the theorem.

We introduce now several characterization of first countability in ZFC, which are not equivalent in general. They are introduce in order to help a better understanding of the possible choice free versions of the previous theorem.

Definitions 6.2.2 Let X be a topological space. We say that X satisfies:

D if $(\forall x)(\forall \mathcal{V}(x)$ open local base at $x)\,(\exists \mathcal{B}(x) \subseteq \mathcal{V}(x))\ |\mathcal{B}(x)| \leq \aleph_0$ and $\mathcal{B}(x)$ is a local base at x;

E if $(\forall (\mathcal{V}(x))_{x\in X}$ with $\mathcal{V}(x)$ open local base at $x)\,(\exists (\mathcal{B}(x))_{x\in X})$ $(\forall x)\mathcal{B}(x) \subseteq \mathcal{V}(x)$, $|\mathcal{B}(x)| \leq \aleph_0$ and $\mathcal{B}(x)$ is a local base at x;

F if $(\forall (\mathcal{V}(x))_{x\in X}$ with $\mathcal{V}(x)$ open local base at $x)\,(\exists \{B(n,x) : n \in \mathbb{N},\ x \in X\})$ $(\forall x)[(\forall n)B(n,x) \in \mathcal{V}(x)$ and $\{B(n,x) : n \in \mathbb{N}\}$ is a local base at $x]$;

G if $(\forall x)(\forall \mathcal{V}(x)$ local base at $x)\,(\exists \mathcal{B}(x) \subseteq \mathcal{V}(x))\ |\mathcal{B}(x)| \leq \aleph_0$ and $\mathcal{B}(x)$ is a local base at x.

H if $(\forall (\mathcal{V}(x))_{x\in X}$ with $\mathcal{V}(x)$ local base at $x)\,(\exists (\mathcal{B}(x))_{x\in X})$ $(\forall x)\mathcal{B}(x) \subseteq \mathcal{V}(x)$, $|\mathcal{B}(x)| \leq \aleph_0$ and $\mathcal{B}(x)$ local base at x;

I if $(\forall (\mathcal{V}(x))_{x\in X}$ with $\mathcal{V}(x)$ local base at $x)\,(\exists \{B(n,x) : n \in \mathbb{N},\ x \in X\})$ $(\forall x)[(\forall n)B(n,x) \in \mathcal{V}(x)$ and $\{B(n,x) : n \in \mathbb{N}\}$ is a local base at $x]$;

Together with the definitions G–I which try to transfer to ZF the characterization of 6.2.1, one includes three other definitions where the given local bases are open.

Proposition 6.2.3 *For the classes* A–I*:*

(a) C ⊆ B ⊆ A*;*

(b) F ⊆ E ⊆ D*;*

(c) I ⊆ H ⊆ G*;*

(d) G ⊆ D ⊆ A*;*

(e) H ⊆ E ⊆ B*;*

(f) I ⊆ F ⊆ C.

$$
\begin{array}{ccccc}
G & \Longleftarrow & H & \Longleftarrow & I \\
\Downarrow & & \Downarrow & & \Downarrow \\
D & \Longleftarrow & E & \Longleftarrow & F \\
\Downarrow & & \Downarrow & & \Downarrow \\
A & \Longleftarrow & B & \Longleftarrow & C
\end{array}
$$

Lemma 6.2.4 *Every topological space with a countable topology satisfies F, and then E and D.*

Theorem 6.2.5 ([16]) *The following conditions are equivalent to* **CC***:*

(i) *if, in a topological space, x has a countable local base, then every local base at x contains a countable local base;*

(ii) *a topological space satisfies* A *if and only if satisfies* G*;*

(iii) *a topological space satisfies* A *if and only if satisfies* D*;*

(iv) *a topological space satisfies* D *if and only if satisfies* G.

Note that condition (ii) is the Theorem 6.2.1.

Corollary 6.2.6 *The following conditions are equivalent to* **CC***:*

(i) *every metric space satisfies* G *(respectively* D*);*

(ii) *every second countable space satisfies* G*;*

(iii) *every space with a countable topology satisfies* G.

Proposition 6.2.7 ([16]) *The following conditions are equivalent:*

(i) $\mathbf{MC}_\omega$ *and* **CC***;*

(ii) $\mathbf{MC}_\omega$ *and* **CUC***;*

(iii) $\mathbf{MC}_\omega$ *and* $\mathbf{CC}(\aleph_0)$*;*

(iv) *a topological space satisfies* B *if and only if satisfies* H*;*

(v) *a topological space satisfies* B *if and only if satisfies* E*;*

(vi) *a topological space satisfies* E *if and only if satisfies* H.

Corollary 6.2.8 *Every first countable space (i.e satisfies* A*) satisfies G if and only if* $\mathbf{MC}_\omega$ *and* **CC** *hold.*

This is a possible alternative, in ZF, to the Theorem 6.2.1.

Corollary 6.2.9 *The following conditions are equivalent to* $\mathbf{MC}_\omega$ *and* **CC***:*

(i) *every metric space satisfies* H *(respectively* E*);*

(ii) *every second countable space satisfies* H*;*

(iii) *every space with a countable topology satisfies* H.

Proposition 6.2.10 *The following conditions are equivalent to the Axiom of Choice:*

(i) *every first countable space (≡*A*) satisfies* I*;*

(ii) *a topological space satisfies* C *if and only if satisfies* I*;*

(iii) *a topological space satisfies* C *if and only if satisfies* F*;*

(iv) *a topological space satisfies* F *if and only if satisfies* I.

The condition (i) of this theorem is other alternative to the Theorem 6.2.1.

Corollary 6.2.11 *The following conditions are equivalent to* **AC***:*

(i) *every metric space satisfies* I *(respectively* F*);*

(ii) *every second countable space satisfies* I*;*

(iii) *every space with a countable base satisfies* I.

It is some how surprising that an apparently so weak condition, such as condition (iii), is equivalent to the Axiom of Choice itself.

Proposition 6.2.12 *If* $\mathbf{MC}_\omega$ *holds, then:*

(a) *a topological space satisfies* D *if and only satisfies* E*;*

(b) *a topological space satisfies* G *if and only satisfies* H.

Corollary 6.2.13 *If the Axiom of Countable Choice holds, then the following conditions are equivalent:*

(i) $\mathbf{MC}_\omega$*;*

(ii) *a topological space satisfies* D *if and only satisfies* E*;*

(iii) *a topological space satisfies* G *if and only satisfies* H.

Proposition 6.2.14 *If* E *is equivalent to* F *or* H *is equivalent to* I*, then* $\mathbf{MC}(\aleph_0)$ *holds.*

6.3 The real numbers

We recall that $\mathbb{R}$ satisfies each of the three definitions A–C.

Theorem 6.3.1 *The following conditions are equivalent to* $\mathbf{CC}(\mathbb{R})$*:*

(i) *every second countable space satisfies* D*;*

(ii) $\mathbb{R}$ *satisfies* D.

Lemma 6.3.2 *If* X *is a SSC (every base contains a countable one)* T_1*-space, then* X *satisfies* D.

At this point, one can see that the Theorem 3.2.2 could easily be proved from the Theorem 6.3.1 and Lemma 6.3.2.

Proposition 6.3.3 *The following conditions are equivalent to the Axiom of Countable Choice for families of subsets of* $\mathcal{P}(\mathbb{R})$ *(*$\mathbf{CC}(\mathcal{P}\mathbb{R})$*):*

(i) *every second countable* T_0*-space satisfies* G*;*

(ii) $\mathbb{R}$ *satisfies* G.

Proposition 6.3.4 *Every second countable space satisfies* F *if and only if the Axiom of Choice holds in* $\mathbb{R}$ *(*$\mathbf{AC}(\mathbb{R})$*).*

Corollary 6.3.5 *The topological space* $\mathbb{R}$ *satisfies* F *if and only if the Axiom of Choice holds for families* $(X_i)_{i\in\mathbb{R}}$ *of non-empty subsets of* $\mathbb{R}$.

Corollary 6.3.6 *The following conditions are equivalent:*

(i) $\mathbf{MC}_\omega(\mathbb{R})$ *and* $\mathbf{CC}(\mathbb{R})$*;*

(ii) *every second countable space satisfies* E.

Corollary 6.3.7 *The following conditions are equivalent:*

(i) $\mathbf{MC}_\omega$ *holds for families* $(X_i)_{i\in\mathbb{R}}$ *of non-empty subsets of* $\mathbb{R}$ *and* $\mathbf{CC}(\mathbb{R})$ *also holds;*

(ii) $\mathbb{R}$ *satisfies* E.

Corollary 6.3.8 *The topological space* $\mathbb{R}$ *satisfies* I *if and only if the Axiom of Choice holds for families* $(X_i)_{i\in\mathbb{R}}$ *of non-empty subsets of* $\mathcal{P}(\mathbb{R})$.

Corollary 6.3.9 *The following conditions are equivalent:*

(i) $\mathbf{MC}_\omega$ *holds for families* $(X_i)_{i\in\mathbb{R}}$ *of non-empty subsets of* $\mathcal{P}(\mathbb{R})$ *and* $\mathbf{CC}(\mathcal{P}\mathbb{R})$ *also holds;*

(ii) $\mathbb{R}$ *satisfies* H.

In the second section of this chapter, we have shown that the Axiom of Choice is a necessary condition to prove that *every space with a countable topology satisfies* I. As we did in other situations, we will look now to the situation of T_0-space with a countable topology. The results are a beat surprising, mainly because they are identical for the classes G, H and I.

Proposition 6.3.10 *The following conditions are equivalent to* $\mathbf{CC}(\mathbb{R})$*:*

(i) *every* T_0*-space with a countable topology satisfies* I*;*

(ii) *every* T_0*-space with a countable topology satisfies* H*;*

(iii) *every* T_0*-space with a countable topology satisfies* G.

Notations

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