Actuator effect of a piezoelectric anisotropic plate model

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Abstract

This paper addresses the actuator effect of a piezoelectric anisotropic plate model, depending on the location of the applied electric potentials, and for different clamped boundary conditions. It corresponds to integer optimization problems, whose objective functions involve the displacement of the plate. We adopt the two-dimensional piezoelectric anisotropic nonhomogeneous plate model derived in Figueiredo and Leal [1]. This model is first discretised by the finite element method. Then, we describe the associated integer optimization problems, which aim to find the maximum mechanical displacement of the plate, as a function of the location of the applied electric potentials. In this sense, we also introduce a related multi-objective optimization problem, which is solved with genetic algorithms. Several numerical examples are reported. For all the tests, the stiffness matrices and force vectors have been evaluated with the subroutines planre and platre, of the CALFEM toolbox of MATLAB [2], and the genetic algorithms have been implemented in C++.

Keywords – Piezoelectric Material, Plate, Finite Elements, Genetic Algorithms.

1 Introduction

Piezoelectric materials are characterized by the interaction between their mechanical and electrical behaviors (cf. Ikeda [3]). In this paper, we analyse the actuator effect of an asymptotic model for an anisotropic piezoelectric thin plate, subjected to the influence of the location of the applied electric potentials and for different clamped boundary conditions. This is an integer optimization problem that we numerically solve by genetic algorithms.

In the literature (both mathematical and engineering) there are several works reporting asymptotic models for anisotropic (or isotropic) piezoelectric thin plates (see for instance Rahmoune, Benjeddou and Ohayon [4], where a survey on this topic is presented). Among those, we refer here some articles, which extend to piezoelectric plates the asymptotic analysis procedure developed formerly by Ciarlet and Destuynder [5] for elastic plates: Rahmoune, Benjeddou and Ohayon [4], and Figueiredo and Leal [1] for heterogeneous, anisotropic and piezoelectric plates (see also Rahmoune [6], and Rahmoune, Osmont, Benjeddou and Ohayon [7], for the numerical implementation of [4]), Sene [8] for homogeneous and isotropic plates, and Raoult and Sene [9] for piezoelectric plates including magnetic effects. In particular, the plate model adopted in this paper was deduced in Figueiredo and Leal [1] (cf. theorem 3.4 in [1]), for

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heterogeneous, anisotropic material with the modified coefficients (which couple the elastic, piezoelectric and dielectric coefficients) constant through the thickness of the plate (see $p_{33\alpha\beta}$ and $p_{33\gamma}$, defined in (3) and (4) in this paper). This adopted model coincides exactly with that derived by Rahmounne, Benjeddou and Ohayon [4], for the case of a short circuited plate (as defined in [4]) and with the same hypotheses on the modified coefficients. We would like to emphasize that in spite of the coincidence of the models and the similarity of the asymptotic development technique used in both papers, Figueiredo and Leal [1] and Rahmounne, Benjeddou and Ohayon [4], there are mainly three differences that distinguish the results of these two papers, and that we shortly describe. Firstly, the mechanical and electric boundary conditions imposed to the three-dimensional plate are not equal (compare (2), (4) in [4] with (10), (11) in the present paper, or with (2), (4) in [1]); in particular in [1] there are mechanical forces and also electric potential applied on the lateral boundary of the three-dimensional plate, hence it is not possible to decompose the electric potential in the same way as in [4] (compare (13) and (23) in [1] with (15) in [4]). Secondly, the scalings of the components of the electric displacements and stresses are different for the two papers (compare (25) and (27) in [4] with (21) in [1]). Finally, theorem 3.3 in [1] (prior to theorem 3.4 which is a particular case of theorem 3.3) exhibits the variational formulation of the asymptotic plate model obtained in [1], when the modified coefficients ($p_{33\alpha\beta}$ and $p_{33\gamma}$, already mentioned) are arbitrary and may depend on the thickness of the plate; a straightforward computation shows that this theorem 3.3 implies that the electric potential depends not only on the bending displacement (as shown in theorem 3.4 of [1], when $p_{33\alpha\beta}$ and $p_{33\gamma}$ are independent of the thickness variable), but also on the tangential displacement of the plate (this result is not found in [4]), and consequently the bending and tangential mechanical displacements are coupled in the asymptotic model.

After this introduction, the rest of the paper is structured as follows. In section 2, the asymptotic piezoelectric plate model is described. Then, we present in section 3 the model's finite element discretization, as well as the optimization problems. In the last two sections we report the numerical results for a particular chosen material (a PZT material, transversely isotropic, with constant piezoelectric and dielectric coefficients), and point out the conclusions and future work.

## 2 The asymptotic piezoelectric plate model

In this section we first introduce some notations concerning the geometry, the material properties, the loadings and boundary conditions imposed on the plate. Then, we recall the static three-dimensional piezoelectric model for a nonhomogeneous anisotropic thin plate and, afterwards, the variational formulation of the asymptotic piezoelectric plate model, deduced by Figueiredo and Leal [1], is described.

### 2.1 Geometry and general notations

Let $O X_1 X_2 X_3$ be a fixed three-dimensional coordinate system, and $\omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz continuous boundary $\partial \omega$ and $\gamma_0$, $\gamma_e$ subsets of $\partial \omega$, such that, $\gamma_0 \neq \emptyset$ and measure$(\gamma_e) \geq 0$. We also define $\gamma_1 = \partial \omega \setminus \gamma_0$, $\gamma_s = \partial \omega \setminus \gamma_e$.

We consider the sets

$$
\Omega = \omega \times (-h, h), \quad \Gamma_{\pm} = \omega \times \{\pm h\}, \quad \Gamma_+ = \omega \times \{+h\}, \quad \Gamma_- = \omega \times \{-h\}, \\
\Gamma_D = \gamma_0 \times (-h, h), \quad \Gamma_1 = \gamma_1 \times (-h, h), \\
\Gamma_N = \Gamma_1 \cup \Gamma_{\pm}, \quad \Gamma_{eN} = \gamma_s \times (-h, h), \quad \Gamma_{eD} = \Gamma_{\pm} \cup (\gamma_e \times (-h, h)),
$$

(1)

where $\overline{\Omega}$ (that is, $\Omega$ and its boundary) represents a thin plate with middle surface $\omega$ and thickness $2h$, with $h > 0$ a small constant, $\Gamma_+$ and $\Gamma_-$ are, respectively, the upper and lower faces of $\Omega$, the sets $\Gamma_D$, $\Gamma_1$ and $\Gamma_{eN}$ are portions of the lateral surface $\partial \omega \times (-h, h)$ of $\Omega$, and finally $\Gamma_N$ and $\Gamma_{eD}$ are portions of the boundary $\partial \Omega$ of $\Omega$. An arbitrary point of $\Omega$ is denoted by $x = (x_1, x_2, x_3)$, where the first two components $(x_1, x_2) \in \omega$ and $x_3 \in (-h, h)$.

Throughout the paper, the latin indices $i, j, k, l \ldots$ belong to the set $\{1, 2, 3\}$, the greek indices $\alpha, \beta, \mu \ldots$ vary in the set $\{1, 2\}$ and the summation convention with respect to repeated indices is employed,
that is, $a_i b_i = \sum_{i=1}^{3} a_i b_i$. Moreover we denote by $a \cdot b = a_i b_i$ the inner product of the vectors $a = (a_i)$ and $b = (b_i)$, by $C e = (C_{ijkl} e_{kl})$ the contraction of a fourth order tensor $C = (C_{ijkl})$ with a second order tensor $e = (e_{kl})$ and by $C d = d = (d_{ij})$ the inner product of the tensors $C e$ and $d = (d_{ij})$.

Given a function $\theta(x)$ defined in $\Omega$ we denote by $\theta_i$ or $\partial_i \theta$ its partial derivative with respect to $x_i$, that is, $\theta_i = \partial_i \theta = \frac{\partial \theta}{\partial x_i}$, and by $\theta_{ij}$ or $\partial_{ij} \theta$ its second partial derivative with respect to $x_i$ and $x_j$, that is, $\theta_{ij} = \partial_{ij} \theta = \frac{\partial^2 \theta}{\partial x_i \partial x_j}$. We denote by $\nu = (\nu_1, \nu_2, \nu_3)$ the outward unit normal vector to $\partial \Omega$, by the same letter $\nu = (\nu_1, \nu_2)$ the outward unit normal vector to $\partial \omega$, and finally by $\partial_{\nu} \theta = \nu_a \partial_a \theta$ the outer normal derivative along $\partial \omega$, of the scalar function $\theta$ defined in $\omega$.

### 2.2 Material

We suppose that a piezoelectric material occupies the bounded thin plate $\Omega \subset \mathbb{R}^3$. We denote by $C = (C_{ijkl})$, $P = (P_{ijk})$ and $\varepsilon = (\varepsilon_{ij})$, respectively, the elastic (fourth-order) tensor field, the piezoelectric (third-order) tensor field, and the dielectric (second-order) tensor field, that characterize the material. The coefficients $C_{ijkl}$, $P_{ijk}$, $\varepsilon_{ij}$ are smooth enough functions defined in $\omega \times [-h, h]$, and they verify the following symmetry properties: $P_{ijk} = P_{kij}$, $\varepsilon_{ij} = \varepsilon_{ji}$, $C_{ijkl} = C_{jikl} = C_{klji}$. In addition, we impose that $C_{\alpha \beta \gamma \rho} = 0 = C_{\alpha 333}$, meaning the material is monoclinic in the plane $OX_1 X_2$, and therefore the number of independent elastic coefficients $C_{ijkl}$ is equal to 13. We also need to introduce the following reduced elastic coefficients

$$A_{\alpha \beta \gamma \rho} = C_{\alpha \beta \gamma \rho} - \frac{C_{\alpha 333} C_{33 \gamma \rho}}{C_{3333}},$$

the modified piezoelectric coefficients $p_{3\alpha \beta}$ and corresponding vector $p_3$

$$p_{3\alpha \beta} = P_{3\alpha \beta} - \frac{C_{\alpha 333} P_{333}}{C_{3333}}, \quad p_3 = [p_{311} \ p_{322} \ p_{312}],$$

and the scalar field $p_{33}$

$$p_{33} = \varepsilon_{33} + \frac{P_{333} P_{33}}{C_{3333}} + \frac{1}{\det \begin{bmatrix} C_{1313} & C_{1323} \\ C_{2313} & C_{2323} \end{bmatrix}} \begin{bmatrix} P_{323} \\ -P_{313} \end{bmatrix}^T \begin{bmatrix} C_{1313} & C_{1323} \\ C_{2313} & C_{2323} \end{bmatrix} \begin{bmatrix} P_{323} \\ -P_{313} \end{bmatrix}. \tag{4}$$

Finally, we also define the following matrices $A$ and $p$, associated to the reduced elastic coefficients $A_{\alpha \beta \gamma \rho}$ and to the modified piezoelectric coefficients $p_{3\alpha \beta}$ and $p_{33}$

$$A = \begin{bmatrix} A_{1111} & A_{1122} & A_{1112} \\ A_{2211} & A_{2222} & A_{2212} \\ A_{1211} & A_{1222} & A_{1212} \end{bmatrix}, \quad p = \begin{bmatrix} p_{311} P_{311} & p_{311} P_{322} & p_{311} P_{312} \\ p_{322} P_{311} & p_{322} P_{322} & p_{322} P_{312} \\ p_{312} P_{311} & p_{312} P_{322} & p_{312} P_{312} \end{bmatrix}. \tag{5}$$

### 2.3 Loadings and boundary conditions

Let $f = (f_i) : \Omega \rightarrow \mathbb{R}^3$ be the density of the applied body forces acting on the plate $\Omega$, $g = (g_i) : \Gamma_N \rightarrow \mathbb{R}^3$ the density of the applied surface forces on $\Gamma_N$ ($g = 0$ in $\Gamma_1$, and $g^+$ and $g^-$ are the restriction of $g$ to $\Gamma^+$ and $\Gamma^-$, respectively). The plate is clamped along $\Gamma_D$, the electric potential applied on $\Gamma_e D$ is represented by $\varphi_0$, and $\varphi_0^+$ and $\varphi_0^-$ denote the restrictions of $\varphi_0$ to $\Gamma_+$ and $\Gamma_-$, respectively. Moreover, there is neither electric charge in $\Omega$ (this means that the material is dielectric) nor on $\Gamma_e N$.

### 2.4 The three-dimensional piezoelectric plate model

In the framework of small deformations and linear piezoelectricity, the three-dimensional static equations for the piezoelectric plate are the following: find a displacement vector field $u : \Omega \rightarrow \mathbb{R}^3$ and an electric
potential $\varphi : \Omega \to \mathbb{R}^3$, such that

\begin{align*}
\sigma &= Ce(u) - PE(\varphi), \quad \text{in } \Omega, \\
D &= Pe(u) + \varepsilon E(\varphi), \quad \text{in } \Omega, \\
\text{div}\sigma &= -f, \quad \text{in } \Omega, \\
\text{div}D &= 0, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \Gamma_D, \quad \sigma \nu = g, \quad \text{on } \Gamma_N, \\
D\nu &= 0, \quad \text{on } \Gamma_{eN}, \quad \varphi = \varphi_0, \quad \text{on } \Gamma_{eD}.
\end{align*}

In (6-11), $\sigma : \Omega \to \mathbb{R}^3$ is the stress tensor field, $D : \Omega \to \mathbb{R}^3$ is the electric displacement vector field, $e(u)$ is the linear strain tensor, defined by

\[ e(u) = (e_{ij}(u)), \quad e_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i), \tag{12} \]

and $E(\varphi)$ is the electric vector field, defined by

\[ E(\varphi) = (E_i(\varphi)), \quad E_i(\varphi) = -\partial_i \varphi. \tag{13} \]

The equations (6-7) are the constitutive equations, (8) is the equilibrium mechanical equation, (9) is the Maxwell-Gauss equation, (10) are the displacement and traction boundary conditions and finally (11) represents the electric boundary conditions.

### 2.5 The space of admissible displacements

This is a Kirchhoff-Love displacement space $V_{KL}$ (that includes boundary conditions) defined by

\[ V_{KL} = \left\{ v : \Omega \to \mathbb{R}^3, \quad v(x) = (v_1(x), v_2(x), v_3(x)), \right. \\
\left. x = (x_1, x_2, x_3) \in \Omega, \right. \\
v_1(x) &= \eta_1(x_1, x_2) - x_3 \partial_1 \eta_3(x_1, x_2), \\
v_2(x) &= \eta_2(x_1, x_2) - x_3 \partial_2 \eta_3(x_1, x_2), \\
v_3(x) &= \eta_3(x_1, x_2), \quad \text{where } \eta = (\eta_i) : \omega \to \mathbb{R}^3, \\
\eta &= (\eta_1, \eta_2, \eta_3) = (0, 0, 0) \quad \text{and } \partial_\nu \eta_3 = 0, \quad \text{in } \gamma_0 \right\}. \tag{14} \]

### 2.6 The asymptotic piezoelectric plate model

This model, derived by Figueiredo and Leal [1] by the asymptotic expansion method, is described by the following formulas (15-19). Briefly it consists of two parts. The first part (i) establishes that the displacement of the plate is a Kirchhoff-Love displacement, and the solution of an equation formulated in the middle plane of the plate, and the second part (ii) defines the exact expression of the electric potential of the plate (it is a second order polynomial with respect to the thickness variable, with coefficients that depend on the transverse component of the Kirchhoff-Love displacement).

(i) The displacement $u : \Omega \to \mathbb{R}^3$ of the plate is a Kirchhoff-Love displacement vector field, that is,

\[ u(x) = (u_1(x), u_2(x), u_3(x)), \quad x = (x_1, x_2, x_3) \in \bar{\Omega}, \]

\[ u_1(x) = \zeta_1(x_1, x_2) - x_3 \partial_1 \zeta_3(x_1, x_2), \]

\[ u_2(x) = \zeta_2(x_1, x_2) - x_3 \partial_2 \zeta_3(x_1, x_2), \]

\[ u_3(x) = \zeta_3(x_1, x_2), \quad \zeta = (\zeta_1, \zeta_2, \zeta_3) = (0, 0, 0) \quad \text{and } \partial_\nu \zeta_3 = 0, \quad \text{on } \gamma_0. \tag{15} \]
and $u$ is the unique solution of the variational problem

$$\text{find } u \in V_{KL} \text{ such that: } a(u, v) = l(v), \quad \forall v \in V_{KL},$$

where

$$l(v) = \int_{\Omega} f \cdot v \, d\Omega + \int_{\Gamma_N} g \cdot v \, d\Gamma_N - \int_{\Omega} \frac{\psi - \varphi}{2h} p_{3\alpha\beta} \epsilon_{\alpha\beta}(v) \, d\Omega,$$

$$a(u, v) = \int_{\omega} \left[ N_{\alpha\beta} \epsilon_{\alpha\beta}(\eta) + M_{\alpha\beta} \partial_{\alpha\beta}\eta_3 \right] \, d\omega,$$

with $N = (N_{\alpha\beta})$ and $M = (M_{\alpha\beta})$, the second-order tensor fields associated to the Kirchhoff-Love displacement $u$, defined by the following matrix formula

$$\begin{bmatrix} N_{\alpha\beta} \\ M_{\alpha\beta} \end{bmatrix} = \begin{bmatrix} \int_{-h}^{+h} A_{\alpha\beta\gamma\rho} \, dx_3 \\ -\int_{-h}^{+h} x_3 A_{\alpha\beta\gamma\rho} \, dx_3 \end{bmatrix} - \begin{bmatrix} \int_{-h}^{+h} x_3^2 (A_{\alpha\beta\gamma\rho} + \frac{p_{3\alpha\beta} p_{3\gamma}}{p_{33}}) \, dx_3 \end{bmatrix} \epsilon_{\gamma\rho}(\zeta_3).$$

(ii) The electric potential $\varphi$ is a second order polynomial in $x_3$, whose coefficients depend on $\zeta_3$, and the exact analytic form of $\varphi$ is the following

$$\varphi(x_1, x_2, x_3) = \frac{\varphi_0^+ + \varphi_0^-}{2} + \frac{\varphi_0^+ - \varphi_0^-}{2h} x_3 + (h^2 - x_3^2) \frac{p_{3\alpha\beta}}{2p_{33}} \partial_{\alpha\beta}\zeta_3.$$

We remark that in order to obtain (18)-(19) it must be assumed that $p_{3\alpha\beta}$ and $p_{33}$ are independent of $x_3$ (cf. theorem 3.4, Figueiredo and Leal [1]). The assumption is satisfied, for example, for a material such that the elastic coefficients $C_{\alpha\beta33}, C_{\alpha3\beta3}, C_{3333}$, the piezoelectric coefficients $P_{3\alpha\beta}, P_{3\alpha3}, P_{333}$ and the dielectric coefficient $\varepsilon_{33}$ are independent of $x_3$.

3 Discrete model

We describe in this section, the approximation of (16) and (19), by the finite element method. Beginning with the matrix formulation of the bilinear form defined in (17), we proceed with the application of the finite element method to the variational formulation (16). The discrete model is completely defined in Theorem 3.1 (see (34) in subsection 3.3). Afterwards, we formulate the integer optimization problems and briefly mention the genetic algorithms that are used to determine the numerical solution of the examples, described in section 4.

3.1 Matrix formulation of the bilinear form

We remark that the bilinear form $a(\cdot, \cdot)$ in (17) can be written

$$a(u, v) = \int_{\omega} V^T B U \, d\omega,$$

where, for any $u$ and $v$ in the space $V_{KL}$

$$V^T = [e_{11}(\eta) \ e_{22}(\eta) \ 2e_{12}(\eta) \ \partial_{11}\eta_3 \ \partial_{22}\eta_3 \ 2\partial_{12}\eta_3],$$

$$U^T = [e_{11}(\xi) \ e_{22}(\xi) \ 2e_{12}(\xi) \ \partial_{11}\xi_3 \ \partial_{22}\xi_3 \ 2\partial_{12}\xi_3].$$
and $B$ is the following matrix of order six (which is symmetric since the sub-matrices $G$, $H$ and $I$ are symmetric)

$$B = \begin{bmatrix} G & -H \\ -H & I \end{bmatrix}_{6 \times 6}$$

where

$$G = \int_{-h}^{+h} A \, dx_3 = \begin{bmatrix} \int_{-h}^{+h} A_{1111} \, dx_3 & \int_{-h}^{+h} A_{1122} \, dx_3 & \int_{-h}^{+h} A_{1112} \, dx_3 \\ \int_{-h}^{+h} A_{2211} \, dx_3 & \int_{-h}^{+h} A_{2222} \, dx_3 & \int_{-h}^{+h} A_{2212} \, dx_3 \\ \int_{-h}^{+h} A_{1211} \, dx_3 & \int_{-h}^{+h} A_{1222} \, dx_3 & \int_{-h}^{+h} A_{1212} \, dx_3 \end{bmatrix},$$

and

$$H = \int_{-h}^{+h} x_3 \, A \, dx_3, \quad I = \int_{-h}^{+h} x_3^2 \, [A + p] \, dx_3.$$

### 3.2 Finite element discretization

A rectangular domain $\omega$ is assumed and it is partitioned into a mesh of $m = n_1 n_2$ sub-rectangles, where $n_1$ is the number of sub-intervals in the $OX_1$ direction and $n_2$ the number of sub-intervals in the $OX_2$ direction. This means that $\omega = \bigcup_{i=1}^{m} \omega^i$, and, for each $i$, $\omega^i = [a_1^i, b_1^i] \times [c_2^i, d_2^i]$. The amplitudes of the real sub-intervals, $[a_1^i, b_1^i]$ and $[c_2^i, d_2^i]$, are denoted by $h_1^i = b_1^i - a_1^i$ and $h_2^i = d_2^i - c_2^i$, respectively. Moreover, we suppose that the mesh $\{\omega^i\}_{i=1,...,m}$ is affine equivalent to the reference element $\tilde{\omega} = (-1, +1) \times (-1, +1)$. The affine transformations are defined by the mappings

$$T^i : \omega^i = [a_1^i, b_1^i] \times [c_2^i, d_2^i] \rightarrow \tilde{\omega} = (-1, +1) \times (-1, +1) : (x_1, x_2) \rightarrow \left( \frac{2}{h_1^i} (x_1 - x_c^i), \frac{2}{h_2^i} (x_2 - y_c^i) \right),$$

where $x_c^i, y_c^i$ are the middle points of $[a_1^i, b_1^i]$ and $[c_2^i, d_2^i]$, respectively, and $(x_1, x_2)$ is a generic element of $\omega^i$.

The rectangular Melosh finite element (cf. subroutine planre of CALFEM [2] and Ciarlet [10]) is considered to approximate the tangential displacement field $(\zeta_1, \zeta_2)$ of the Kirchhoff-Love displacement $u$ defined in (15); the 8 degrees of freedom of the Melosh element are the values of $(\zeta_1, \zeta_2)$ at each vertex of the element $\omega^i$. The four shape functions of the Melosh finite element, defined in $\tilde{\omega}$, are denoted by $M_1, M_2, M_3, M_4$ (the lower subscript indicates the number of the vertex). In the sequel, we use also the matrix $M$ of order $2 \times 8$, defined by

$$M = \begin{bmatrix} M_1 & 0 & M_2 & 0 & M_3 & 0 & M_4 & 0 \\ 0 & M_1 & 0 & M_2 & 0 & M_3 & 0 & M_4 \end{bmatrix}_{2 \times 8}.$$

The Adini finite element (cf. subroutine platre of CALFEM [2] and Ciarlet [10]) is used for the approximation of the transverse displacement $u_3 = \zeta_3$; the 12 degrees of freedom characterizing this element are the values of $u_3, u_{3,1}$ and $u_{3,2}$ at each vertex of $\omega^i$. The twelve shape functions associated to the Adini finite element, defined in $\tilde{\omega}$, are denoted by $N_1^j, N_2^j, N_3^j, N_4^j$, with $j = 1, 2, 3$ (the lower subscript indicates the number of the vertex and the upper subscript $j$ refers to the order of derivation). In the sequel, we also need to introduce the following vector $N^e$, associated to $\omega^i$ and to the twelve shape functions of the Adini finite element

$$N^e = \begin{bmatrix} N_1^i & N_2^i & N_3^i & N_4^i \end{bmatrix}_{1 \times 12}, \quad N_1^i = \frac{1}{2} (M_1 + M_2), \quad N_2^i = \frac{1}{2} (M_2 + M_3), \quad N_3^i = \frac{1}{2} (M_3 + M_4), \quad N_4^i = \frac{1}{2} (M_4 + M_1),$$

and

$$N_1^i = \begin{bmatrix} N_1^i & N_2^i & N_3^i \end{bmatrix}_{1 \times 3}, \quad i = 1, 2, 3, 4.$$

**Remark 3.1.** - The choice of these finite elements is motivated by the available finite elements in CALFEM [2], which is the software used in section 4, for the numerical tests. The rectangular Melosh finite element is conforming and of class $C^0$, and the Adini finite element is also of class $C^0$, but nonconforming (cf. Ciarlet [10], pages 57, 64, 95 and 364-366, respectively). Of course other finite elements could have been used, namely both conforming with advantages for theoretical error estimates proofs. We also notice that
the finite element discretization, explained in the rest of this section 3.2, also applies (with convenable adaptations) to other finite elements choices.

For any Kirchhoff-Love displacement $u$, the tangential displacements $(\zeta_1, \zeta_2)$ and the transverse displacement $u_3 = \zeta_3$ are approximated, at each finite element $\omega^e$, by the following sums

$$
(\zeta_1, \zeta_2)_{|\omega^e} (x_1, x_2) \approx \sum_{i=1}^{4} \left( u_{1i}^e M_i \circ T^e (x_1, x_2), u_{2i}^e M_i \circ T^e (x_1, x_2) \right),
$$

$$
u_3 = \zeta_3_{|\omega^e} (x_1, x_2) \approx \sum_{i=1}^{4} \left( z_{1i}\frac{h}{2}N_1^2, z_{1i}\frac{h}{2}N_1^2, z_{2i}\frac{h}{2}N_1^2 \right) \circ T^e (x_1, x_2),
$$

(28)

where the coefficients $u_{1i}^e$, $u_{2i}^e$ and $z_{1i}^e$, $z_{2i}^e$ are the approximated values of $\zeta_1$, $\zeta_2$ and $\zeta_3$, respectively, at node $i$ of $\omega^e$. Moreover, we denote by $\nu_{1g}^e$ and $\nu_{2g}^e$, the vectors with eight and twelve components, respectively, that approximate, in $\omega^e$, the tangential and transverse displacements $(\zeta_1, \zeta_2)$ and $u_3 = \zeta_3$, that is,

$$
u_{1g}^e = \left[ (u_{11}^e, u_{21}^e)_{i=1}, (u_{12}^e, u_{22}^e)_{i=2}, (u_{13}^e, u_{23}^e)_{i=3}, (u_{14}^e, u_{24}^e)_{i=4} \right]^T \approx (\zeta_1, \zeta_2)_{|\omega^e},
$$

$$
u_{2g}^e = \left[ (z_{11}^e, z_{21}^e)_{i=1,2,3,4} \right]^T \approx \zeta_3_{|\omega^e}.
$$

(29)

In addition, the following vector, with 20 components, is introduced

$$u^e = \begin{bmatrix} u_{1g}^e \\ u_{2g}^e \end{bmatrix}_{20 \times 1},
$$

(30)

that is the local finite element approximation of the displacement vector field $(\zeta_1, \zeta_2, \zeta_3)$ in $\omega^e$.

In order to describe the discrete problem, corresponding to (16) and (19), some further notation must be detailed, concerning the numeration of the global degrees of freedom and nodes in the mesh. So, let $n$ be the number of global nodes, $m$ the number of finite elements defined before, and

$$u_{1g} = \left[ u_{1g}^e \right]_{e=1}^m \in \mathbb{R}^{3n}, \quad u_{2g} = \left[ u_{2g}^e \right]_{e=1}^m \in \mathbb{R}^{2n},
$$

be the global approximations of the transverse and tangential displacements $(\zeta_3)$ and $(\zeta_1, \zeta_2)$, respectively. Thus, let $u \in \mathbb{R}^{5n}$ be the global approximation of $(\zeta_1, \zeta_2, \zeta_3)$ in $\omega$, defined by

$$u = \left[ u_{1g} \ u_{2g} \right] \in \mathbb{R}^{2n+3n},$$

with

$$u_{1g} = \left( u_{1j}, u_{2j} \right)_{j=1}^n, \quad u_{2g} = \left( z_j, z_{1j}, z_{2j} \right)_{j=1}^n.
$$

(31)

(32)

Moreover, the following subsets of indices are defined

$$I_1, I_2, \quad I_1 \cup I_2 \subset \{1, 2, ..., 2n\},
$$

$$J_1, J_2, J_3, \quad J_1 \cup J_2 \cup J_3 \subset \{2n+1, ..., 5n\}.
$$

(33)

The two sets of indices $I_1$ and $I_2$ represent the number of the global degrees of freedom, that are attached to the values of the tangential displacements $\zeta_1$ and $\zeta_2$, respectively, at the boundary nodes of the mesh, where the plate is clamped. Analogously, the three sets $J_k$, for $k = 1, 2, 3$, represent the number of the global degrees of freedom, associated to the transverse displacement $u_3 = \zeta_3$ at the boundary nodes of the mesh, where the plate is clamped: the subscript $k = 1$ refers to the displacement, $k = 2$ to the first derivative of the displacement with respect to $x_1$, and $k = 3$ to the first derivative of the displacement with respect to $x_2$. Moreover, if $J$ is an arbitrary set of indices, and $u_{1g} \in \mathbb{R}^{3n}$, $u_{2g} \in \mathbb{R}^{2n}$, we denote by $u_{1gJ}$, $u_{2gJ}$ the sub-vectors of $u_{1g}$ and $u_{2g}$ respectively, whose components have their indices in $J$.
3.3 Discrete model

Based on the choice of the finite elements described before and using the notations introduced in (33), the following result is obtained.

Theorem 3.1 The discrete problem associated to (16) takes the following form:

\[
\left\{ \begin{array}{l}
\text{Find } u = [u_{t\theta}, u_{tv}] \in \mathbb{R}^{\delta n} \text{ such that:} \\
u_{t\theta} = u_{t\theta} = 0, \\
u_{tv} = u_{tv} = 0,
\end{array} \right. 
\]

(34)

At the element level, the square matrix \( K \) and the vector \( F \) are defined respectively, by \( K^e \) in (40) and \( F^e \) in (44). Furthermore, the finite element approximation of the electric potential (19) is defined by

\[
\phi(x_1, x_2, x_3) \big|_{\omega^e} \simeq \frac{\phi_0 + \phi_0}{2}\frac{\phi_0 - \phi_0}{2h} x_3 + \frac{1}{2p_{33}} [h^2 - x_3^2] p_3 S^e u^e_{tv},
\]

(35)

where \( S^e \), defined in (39), is the matrix of the second derivatives of the Adini’s finite element shape functions.

Proof: The discretisation of (16) can be obtained, directly, by replacing in (20) \((\zeta_1, \zeta_2, \zeta_3)\) and \((\eta_1, \eta_2, \eta_3)\) by the approximations defined in (28). In fact, for any \( u \) and \( v \) in \( V_{KL} \)

\[
a(u, v) = \sum_{e=1}^{m} \int_\omega V^T B U \, d\omega^e = \sum_{e=1}^{m} h^e_1 h^e_2 \int_\omega V^T B U \, d\omega,
\]

(36)

and due to (28), at each finite element \( \omega^e = T^e(\hat{\omega}) \), we can use the approximations

\[
V \simeq \begin{bmatrix} L^e v^e_{tg} \\ S^e v^e_{tv} \end{bmatrix}, \quad U \simeq \begin{bmatrix} L^e u^e_{tg} \\ S^e u^e_{tv} \end{bmatrix},
\]

(37)

where \( L^e \) and \( S^e \) are the two matrices, that depend on the derivatives of the shape functions of the Melosh and Adini’s finite elements, respectively. The matrix \( L^e \) has order \( 3 \times 8 \) and is defined by

\[
L^e = [L^e_1 L^e_2 L^e_3 L^e_4]_{3 \times 8}, \quad L^e_i = \begin{bmatrix} 2\frac{M_{i,1}}{h_1^e} & 0 \\ 0 & 2\frac{M_{i,2}}{h_2^e} \\ \frac{2}{h_1^e} M_{i,1} & \frac{2}{h_2^e} M_{i,2} \end{bmatrix}, \quad i = 1, 2, 3, 4,
\]

(38)

and \( S^e \) is a matrix of order \( 3 \times 12 \) defined by

\[
S^e = [S^e_1 S^e_2 S^e_3 S^e_4]_{3 \times 12}, \quad S^e_i = \begin{bmatrix} \frac{4}{h_1^e h_2^e} & \frac{h_1^e}{2} N^1_{i,11} & \frac{h_1^e}{2} N^1_{i,22} \\ \frac{h_2^e}{2} N^2_{i,11} & \frac{h_2^e}{2} N^2_{i,22} & \frac{h_2^e}{2} N^2_{i,11} \\ \frac{2}{h_1^e h_2^e} & \frac{h_1^e}{2} N^3_{i,11} & \frac{h_1^e}{2} N^3_{i,11} \end{bmatrix}, \quad i = 1, 2, 3, 4.
\]

(39)

Now introducing (37) and (38-39) in (36) we get immediately

\[
a(u, v) \simeq \sum_{e=1}^{m} h^e_1 h^e_2 \int_\omega \left( \begin{bmatrix} L^e v^e_{tg} \\ S^e v^e_{tv} \end{bmatrix}^T B \begin{bmatrix} L^e u^e_{tg} \\ S^e u^e_{tv} \end{bmatrix} \right) \, d\omega^e
\]

\[
= \sum_{e=1}^{m} v^e T \begin{bmatrix} h^e_1 h^e_2 & 0 \\ 0 & S^e T \end{bmatrix}_{20 \times 6} B_{6 \times 6} \begin{bmatrix} L^e & 0 \\ 0 & S^e \end{bmatrix}_{6 \times 20} \, d\omega^e u^e
\]

\[
= \sum_{e=1}^{m} v^e T K^e u^e.
\]

(40)
On the other hand, the linear form \( l(v) \) in (17) can be written

\[
l(v) = \int_{\Omega} f \cdot v \, d\Omega + \int_{\Gamma_N} g \cdot v \, d\Gamma_N - \int_{\Omega} \frac{\varphi_0^+ - \varphi_0^-}{2h} \, p_{3\alpha\beta} e_{\alpha\beta}(v) \, d\Omega
\]

\[
= \frac{1}{4} \sum_{e=1}^{m} h_e^2 h_e^2 \left[ \int_{\Omega} \left( \int_{-h}^{+h} f_\alpha \, dx_3 + g_\alpha^+ + g_\alpha^- \right) (\eta_\alpha - x_3 \partial_3 \eta_\alpha) \, d\omega \right]

+ \int_{\Omega} \left( \int_{-h}^{+h} f_3 \, dx_3 + g_3^+ + g_3^- \right) v_3 \, d\omega + \int_{\Omega} \frac{\varphi_0^+ - \varphi_0^-}{2h} \left( \int_{-h}^{+h} p_{3\alpha\beta} e_{\alpha\beta}(v) \, dx_3 \right) \, d\omega.
\]

But for any \( v \) in \( V_{KL} \), \( p_{3\alpha\beta} e_{\alpha\beta}(v) = p_{3\alpha\beta}[e_{\alpha\beta}(\eta) - x_3 \partial_3 \eta_3] \), and due to (28), the following approximations can be used, in each finite element \( \omega^e = T^e(\hat{\omega}) \),

\[
(\eta_1, \eta_2) \approx M v_{tg}, \quad v_3 = \eta_3 \approx N^e v_{te}, \quad p_{3\alpha\beta} e_{\alpha\beta}(v) \approx p_3 [L^e v_{tg}^e - x_3 S^e v_{te}^e]_{3 \times 1}.
\]

So, denoting

\[
f_{tg} = [F_1 \ F_2]^T \quad \text{and} \quad f_{tv} = [F_3]
\]

and assuming, to simplify the following computations, that \( f_\alpha, g_\alpha^+ \) and \( g_\alpha^- \) are independent of \( x_3 \in (-h, h) \), we obtain that \( l(v) \) is approximated by

\[
l(v) \approx \sum_{e=1}^{m} \left\{ \left( v_{tg}^e \right)^T \frac{h_e^2 h_e^2}{4} \int_{\hat{\omega}} \left[ M^T f_{tg} + \frac{\varphi_0^+ - \varphi_0^-}{2h} \left( \int_{-h}^{+h} L^e T^e p_3^e \, dx_3 \right) \right] \, d\omega \right. \\

+ \left( v_{te}^e \right)^T \frac{h_e^2 h_e^2}{4} \int_{\hat{\omega}} \left[ N^e T^e f_{tv} + \frac{\varphi_0^+ - \varphi_0^-}{2h} \left( \int_{-h}^{+h} x_3 S^e T^e p_3^e \, dx_3 \right) \right] \, d\omega \left\} = \sum_{e=1}^{m} v^e T^e e^e,
\]

where \( F^e = \left[ \begin{array}{c} F_{tg}^e \\ F_{tv}^e \end{array} \right] \) is a vector with 20 components. Therefore, from (40) and (44), we conclude that the asymptotic variational model (16) is approximated by the linear equation

\[
\sum_{e=1}^{m} v^e T^e e^e = \sum_{e=1}^{m} v^e T^e F^e,
\]

which consequently implies the equation \( Ku = F \) in (34). The matrix \( K \), of order \( 5n \), and the vector \( F \), also of order \( 5n \), are obtained by assembling the element matrices \( K^e \) and vectors \( F^e \), by the usual finite element procedure. The components of the unknown \( u \) have the form described in (31-32).

Finally, to obtain (35) it is enough to use (19) and to remark that, in each finite element \( \omega^e = T^e(\hat{\omega}) \), we can use the approximation in the right hand side of (42) (which is a straightforward consequence of (37)) and the definition of the vector \( p_3 \) in (3). Thus we have

\[
p_{3\alpha\beta} \partial_3 \eta_3 |_{\omega^e} \approx p_3 S^e u_{te}^e. \quad \Box
\]

### 3.4 Optimization problems

We describe now the integer optimization problems, that model the actuator effect of the discrete piezoelectric anisotropic plate, as a function of the position of a fixed number of electrodes, through which
the electric potential is applied. The electrodes are stuck on some parts or on the whole upper or/and lower faces of the plate, and they are considered very thin and very light, such that, their mechanical properties are neglected. In addition we suppose the area occupied by each electrode is the area of one finite element of the mesh $\omega = \bigcup_{e=1}^{m} \omega^e$, and at each electrode the applied electric potential is equal to $\varphi_0^e$, if the electrode is on the top of the plate, and to $\varphi_0^e$, if the electrode is on the bottom.

Let us introduce, for a mesh with $m$ finite elements, the new integer variables $y_i = (i, pe)$, $i = 1, ..., m$, where the first component represents the total number of projections of the electrodes in $\omega$, the middle plane of the plate, and the second component $pe$ denotes the location of these projections in the mesh. Thus $pe$ is a subset of $Y = \{1, 2, \ldots, m\}$, with $i$ elements, that is, $\# pe = i$, where $\#$ represents the cardinal of a set. We remark that $pe$ ranges all the subsets of $Y$ with $i$ distinct elements, that is, $pe \in C^m_i(Y)$. For example, for $i = 3$ and $y_3 = (3, [1, 5, 10])$, it means that, the applied electric potential $\varphi_0^- = \varphi_0^+$ are zero everywhere on the mesh $\omega = \bigcup_{e=1}^{m} \omega^e$, except at the finite elements 1, 5, 10, where these two applied electric potentials can not be simultaneously zero, and three situations can occur:

- if both $\varphi_0^- \neq 0 \neq \varphi_0^+$ at the finite elements 1, 5, 10, there is a total of $2 \times 3$ electrodes on the plate, 3 at the top and 3 at the bottom, whose orthogonal projections, on the middle plane of the plate, are the finite elements 1, 5, 10,
- if $\varphi_0^- \neq 0$ and $\varphi_0^+ = 0$ (respectively $\varphi_0^+ \neq 0$ and $\varphi_0^- = 0$), at the finite elements 1, 5, 10, then, there are 3 electrodes located at the lower face (respectively upper face) of the plate, and whose orthogonal projections on the middle plane of the plate are the finite elements 1, 5, 10.

For a mesh with $m$ finite elements, $n$ global nodes, and $i$ orthogonal projections of electrodes (where $1 \leq i \leq m$), the mechanical displacement of the plate is determined by the displacements $(\zeta_1, \zeta_2, \zeta_3)$ of the nodes in the middle plane. But at each node $j$ in the mesh, the corresponding three-dimensional displacement $(\zeta_1, \zeta_2, \zeta_3)$ is approximated by $(u_{1j}, u_{2j}, z_j)$ (cf. (32)). For fixed applied mechanical forces and boundary conditions, the node displacements depend on the number and location of the projections of the electrodes, represented by the optimization variable $y_i = (i, pe)$. Of course, for each fixed $y_i$ there exists always a node in the mesh, which attains a maximum displacement $d_i(y_i)$, that is

$$d_i(y_i) = \max_{j=1,\ldots,n} \|(u_{1j}, u_{2j}, z_j)\|_{\mathbb{R}^3}$$

(47)

and where $\| . \|_{\mathbb{R}^3}$ is the usual Euclidean norm in $\mathbb{R}^3$. The objective is to maximize $d_i(y_i)$, when $i$ is fixed and $pe$ ranges all the subsets of $Y$ with $i$ distinct elements. Thus, for each $i$, with $1 \leq i < m$, the optimization problems that we address in this paper are the following:

$$\max_{y_i} \max_{d_i(y_i)} \left( \max_{j=1,\ldots,n} \|(u_{1j}, u_{2j}, z_j)\|_{\mathbb{R}^3} \right),$$

subject to:

$$y_i = (i, pe), \quad pe \in C^m_i(Y), \quad \# pe = i,$$

(48)

As mentioned before $F$ depends on $y_i$, cf. (44), thus we set $F_{y_i}$ instead of $F$, to emphasize this dependence. Therefore for each $y_i$, the vector $u$ depends on $y_i$, and consequently this optimization problem (48) has the following interpretation: for fixed boundary conditions and mechanical loadings, the aim is to determine, the location $pe$ of the $i$ projections of the electrodes, that cause a maximum node’s displacement in the plate.

It should be referred that this is a combinatorial problem, since different combinations of the positions of the projections of the electrodes can produce different displacements. In particular, the set $C^m_i(Y)$, that is the admissible set for the optimization variable $pe$, has cardinal equal to the combinations of $m$,
\[ C_{im}^m = \frac{m!}{n!(m-n)!} \] (for instance for a mesh with 25 finite elements and 3 electrodes, we have \( C_{25}^3 = 2300 \)). Nevertheless the number \( C_{im}^m \) can be reduced if the problem has some symmetry.

Obviously, the solutions of these optimization problems strongly depend on the mechanical loadings and the boundary conditions imposed to the plate. In order to achieve a better understanding of the actuator effect, it can be assumed that all the mechanical loadings \( f = (f_i) \) and \( g = (g_i) \) are zero. To analyze the influence of the clamped boundary conditions, it may be considered that the plate is clamped on different parts of the lateral surface (this means that we vary the definition of the set \( \gamma_0 \subset \partial \omega \)).

We also remark that problem (48) is a single optimization problem, since there is only one objective, and the purpose is to determine the global optimum solution. But, we are interested in finding the optimum location of the projections of the electrodes, that generate a maximum node displacement, as well as the minimum number of these electrodes (can this minimum number be less then \( m^2 \))? Therefore, two objectives can thus be considered: the maximization of the displacements of any of the nodes of the mesh and the minimization of the number of the electrodes projections. This corresponds to the following multi-objective problem (associated to (48))

\[
\begin{align*}
\max d_i(y_i) & \land \min i \\
\text{subject to :} & \\
& y_i = (i, pe), \quad pe \in C_{im}^m(Y), \quad \#pe = i, \quad i = 1, 2, ..., m, \\
& \text{Find } u = [u_{tg}, u_{tv}] \in \mathbb{R}^{5n} \text{ such that :} \\
& u_{tg_{f_1}} = u_{tg_{f_2}} = 0, \quad u_{tv_{f_1}} = u_{tv_{f_2}} = u_{tv_{f_3}} = 0, \\
& Ku = F_{y_i} \quad (\text{where } F_{y_i} = F, \text{ defined in } (44))
\end{align*}
\]

where \( m \) is the total number of finite elements in the mesh. For this last problem the aim is to characterize, not the optimal solution, but a set of optimal solutions, the so-called set of Pareto optimal solutions; these are solutions that can not improve the performance of the first objective function (the node’s displacement \( d_i(y_i) \)), without making worse the performance of the second one (the number \( i \) of projections of the electrodes).

### 3.5 Genetic Algorithms

Solving multi-objective (engineering) problems is a very difficult task since, in general, the objectives conflict across a high-dimensional problem space (for example – cf. Costa et al. [11] – consider a structural optimization problem, concerning the stiffness of a linearly laminated elastic plate for which the optimization variables are the thickness and the material of each lamina, with several additional constraints involving the global thickness, price, and mass of the plate). Genetic algorithms (GAs) (cf. Goldberg [12]) are particularly suited to tackle this class of problems because they work with populations of candidate solutions, and use some diversity-preserving mechanisms, that enable to find, in a single run, widely different multiple potential Pareto-optimal solutions (cf. Deb [13]).

The Elitist GA, described in Costa and Oliveira [14] and Costa et al. [11], was applied in section 4 to problems (48) and (49), with standard values for the parameters. Next, we briefly describe some technical features and the parameters of this GA.

The optimization variables \( y_i = (i, pe) \) in problems (48) and (49) were encoded using binary strings, referred as chromosomes. For example, consider a fixed mesh with \( m \) finite elements and \( n \) nodes: a binary string represents the sequence of the \( m \) finite elements in the mesh, as well as the position \( pe \) of \( i \) electrodes – 1 means that, the respective finite element is the projection of one electrode, where it has been applied an electric potential, while 0 means that, for the corresponding finite element, there is no projection of an electrode. To each string it is assigned a displacement \( u \), that is the solution of the inner linear system \( Ku = F_{y_i} \), in problems (48) and (49). We recall that \( u \) is a vector containing the displacements of all the \( n \) nodes of the mesh. The aim of the multi-objective problem (49) is the maximization of the displacement of any of the \( n \) nodes of the mesh, as a function of the projection’s location of the electrodes in the mesh, as well as the minimization of the number of these projections.

The stopping criterion of the GA has varied according to the size \( m \) of the finite element mesh: for example for the meshes with 3x3, 4x4 and 5x5 finite elements, the maximum number of generations
allowed was 30, 50 and 100, respectively, and the number of binary decision variables (the chromosomes) was, respectively, 9, 16 and 25. For all the meshes, we have used an initial population size of 100 chromosomes. A tournament selection, a two point crossover and an uniform mutation were adopted. The crossover probability was, for all the meshes, 0.7. The mutation probability was given by $1/b$ where $b$ is the binary string length. The elitism level considered was 10. The value of sigma share ($\sigma_{\text{share}}$) was kept constant for all the meshes and equal to 1. For sharing purposes, the distance measure considered was the Hamming distance between chromosomes.

4 Numerical tests

In this section we report several experiments. For all the tests, the stiffness matrices $K$ and force vectors $F$ have been evaluated with the subroutines $\text{planre}$ and $\text{platre}$, of the CALFEM toolbox of MATLAB [2], and, the genetic algorithms have been implemented in $C^\star$.

In all the numerical tests presented in this section, we have supposed that the reduced elastic coefficients $A_{\alpha\beta\gamma\rho}$ are independent of the thickness variable $x_h$. This assumption clearly simplifies the linear system $K u = F$ in the optimization problem (48). In fact, this implies that matrix $H$ defined in (24) vanishes, and thus the element stiffness matrix $K^e$ (40) reduces to

$$K^e = \frac{b^2 b^2}{4} \int_\Delta \left[ \begin{array}{cc} L^e T G^e & 0 \\ 0 & S^e T I S^e \end{array} \right]_{20 \times 20} d\omega^e = \left[ \begin{array}{c} K_{tg}^e \\ K_{tv}^e \end{array} \right].$$

Hence, the system (45) is equivalent to the two following independent linear systems

$$\sum_{e=1}^m v^e_{tg} K^e_{tg} v^e_{tg} = \sum_{e=1}^m v^e_{tg} F^e_{tg}$$

and

$$\sum_{e=1}^m v^e_{tv} K^e_{tv} v^e_{tv} = \sum_{e=1}^m v^e_{tv} F^e_{tv}.$$ 

Let us now denote by $K_{tg}$ and $K_{tv}$ the square matrices of order $2n$ and $3n$, defined, at the element level, by respectively, $K^e_{tg}$ and $K^e_{tv}$, and denote by $F_{tg}$ and $F_{tv}$ the vectors of order $2n$ and $3n$, defined, at the element level, by respectively, $F^e_{tg}$ and $F^e_{tv}$. Then, from (51), we conclude that the system in (48) is equivalent to the two independent linear systems

$$\begin{cases}
\text{Find } u_{tg} \in \mathbb{R}^{2n} : \\
 u_{tg I_1} = u_{tg I_2} = 0, \quad \text{and} \\
 K_{tg} u_{tg} = F_{tg},
\end{cases}$$

$$\begin{cases}
\text{Find } u_{tv} \in \mathbb{R}^{3n} : \\
 u_{tv J_1} = u_{tv J_2} = u_{tv J_3} = 0, \\
 K_{tv} u_{tv} = F_{tv},
\end{cases}$$

whose unknowns are the tangential and transverse displacement, $u_{tg}$ and $u_{tv}$, respectively. It should be added that the system, on the left hand side, depends on $\phi_0$ or/and $\varphi_0$, though $F_{tg}$, but the system, on the right hand side, is independent of these electric potential data. Thus, the unknown $u_{tg}$ depends on $\phi_0$ or/and $\varphi_0$, but $u_{tv}$ is independent of these data. Therefore, the optimization problem (48) reduces to

$$\begin{cases}
\max_{y_i} d_i(y_i) = \max_{y_i} \left( \max_{j=1, \ldots, n} \| (u_{1j}, u_{2j}) \|_{\mathbb{R}^2} \right), \\
 y_i = (i, p e), \quad p e \in C^m_i(Y), \quad \# p e = i,
\end{cases}
$$

subject to:

$$\begin{cases}
\text{Find } u_{tg} \in \mathbb{R}^{2n} \text{ such that :} \\
 u_{tg I_1} = u_{tg I_2} = 0, \\
 K_{tg} u_{tg} = F_{tg},
\end{cases}$$

and the linear system on the right hand side in (52) can be solved independently of the optimization problem, because it is independent of the integer variable $y = (i, p e)$.

We consider now a fixed three-dimensional coordinate system $OXYZ$, and a plate $\Omega = [0, L_1] \times [0, L_2] \times [-h, +h]$, with a rectangular middle plane $\omega = (0, L_1) \times (0, L_2)$, whose sides have length $L_1$, $L_2$, and thickness $2h$. The geometric, electric potential and mechanical loadings data, imposed to the plate,
that the chosen material is homogeneous and transversely isotropic in the plane. The definition (54) and Table 2 state that the material is piezoelectric with constant piezoelectric and dielectric coefficients (cf. Table 1 and formulas (45), (46), (47) in [15]).

\[
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1222} & C_{1233} & C_{1333} \\
C_{2211} & C_{2222} & C_{2233} & C_{2322} & C_{2333} & C_{3333} \\
C_{3311} & C_{3322} & C_{3333} & C_{3233} & C_{3222} & C_{2222} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
P_{111} & P_{112} & P_{113} & P_{121} & P_{122} & P_{123} \\

P_{211} & P_{212} & P_{213} & P_{231} & P_{232} & P_{233} \\

P_{311} & P_{312} & P_{313} & P_{331} & P_{332} & P_{333} \\
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & 0 & P_{115} & 0 \\
0 & 0 & 0 & P_{115} & 0 & 0 \\
0 & P_{115} & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(54)

\[
\begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{22} & \varepsilon_{23} & \varepsilon_{33} \\
\end{bmatrix}
= \begin{bmatrix}
\varepsilon_{11} & 0 & 0 \\
0 & \varepsilon_{11} & 0 \\
0 & 0 & \varepsilon_{33} \\
\end{bmatrix}
\]

In Tables 1-2 the unit symbols m, V, N, GPa, Cm$^{-2}$ and Fm$^{-1}$ mean, respectively, meter, volt, newton, giga pascal, coulomb per square meter and farad per meter. The definition (54) and Table 2 state that the chosen material is homogeneous and transversally isotropic in the plane $OXY$, with constant piezoelectric and dielectric coefficients.

Regarding the geometric data given in Table 1, we remark that for brittle ceramics the practical dimensions of a PZT patch side do not exceed 7cm; therefore for rough finite element meshes it would be more realistic to consider $L_1 \leq 1m$ and $L_2 \leq 1m$, since each electrode’s projection in the middle plane $\omega$ is equal to a finite element of $\omega$ (but the conclusions presented in Tables 5 and 6, and concerning the location of the electrodes for $3 \times 3$, $4 \times 4$ and $5 \times 5$ finite elements, would be essentially the same).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>$L_1$</td>
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</tr>
<tr>
<td>$L_2$</td>
<td>m</td>
<td>1</td>
</tr>
<tr>
<td>$h$</td>
<td>m</td>
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</tr>
<tr>
<td>$\varphi_0$</td>
<td>V</td>
<td>-100</td>
</tr>
<tr>
<td>$f = (f_i)$</td>
<td>N</td>
<td>(0,0,0)</td>
</tr>
<tr>
<td>$g = (g_i)$</td>
<td>N</td>
<td>(0,0,0)</td>
</tr>
</tbody>
</table>

Table 1: Geometric, electric potential and mechanical loadings data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit</th>
<th>Value</th>
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Table 2: Elastic, piezoelectric and dielectric data

In the sequel, and for each finite element mesh of $\omega$, the finite elements and the nodes are numbered from the left side $ls = \{0\} \times [0,L_2]$ to the right side $rs = \{L_1\} \times [0,L_2]$ and from the bottom side $bs = [0,L_1] \times \{0\}$ to the top side $ts = [0,L_1] \times \{L_2\}$ of $\omega$, as explained in Tables 3 and 4.

The solutions produced by the genetic algorithms are displayed in Tables 5 and 6, for two groups of clamped boundary conditions, for three different meshes of the middle plane $\omega$ (respectively with $3 \times 3$, $4 \times 4$, and $5 \times 5$ finite elements).
Table 3: Finite element numeration of the meshes with 3x3, 4x4 and 5x5 finite elements

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Table 4: Node numeration for the meshes with 3x3, 4x4 and 5x5 finite elements

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</table>

4 × 4 and 5 × 5 finite elements) and for 1 to 5 projections of electrodes. In these tables, \(NFE\) is the number of finite elements of the mesh, \(ne\) is the number of projections of electrodes, \(pe\) is the optimal positions of these projections in the mesh, \(N\) is the number of the global node of the mesh where the maximum displacement is attained, and \(mv\) represents the value of this maximum displacement, in meters and multiplied by the scalar \(10^6\) (for example, if \(ne = i\) and the optimal location of the projections are defined by \(pe\), then \(mv = d_i(y_i)\) with \(y_i = (i,pe)\)). Finally \(BC\) denotes the type of clamped boundary conditions. If \(BC = 1\), \(\omega\) is clamped only on the bottom side \((\gamma_0 = bs)\); if \(BC = 2\), \(\omega\) is clamped on the left, bottom and right sides \((\gamma_0 = ls \cup bs \cup rs)\); if \(BC = 3\), \(\omega\) is clamped on the two opposite left and right sides \((\gamma_0 = ls \cup rs)\); if \(BC = 4\), \(\omega\) is clamped on the two consecutive bottom and right sides \((\gamma_0 = bs \cup rs)\).

A direct observation of Tables 5 and 6 leads to the following conclusions. For each type of clamped boundary conditions and for each fixed number of electrodes, there is always more than one solution \(pe\), except for the case \(BC = 4\) and \(ne = 1\). Moreover, these multiple solutions correspond to symmetric positions of the projections of the electrodes in the mesh. These results are physical meaningful since the middle plane \(\omega\) is a square and the finite element meshes are regular and square (3x3, 4x4, 5x5).

The Tables 5 and 6 also show that a refinement of the mesh clearly defines the optimal location of the projections of the electrodes (see Table 7 that also illustrates this fact); the corresponding nodes \(N\), where the displacements are maximum, are precisely those nodes that are far away from the clamped sides. We also conclude, that, among the four boundary conditions, the case \(BC = 1\) originates larger displacements than the other 3 cases.

Figure 1 displays the undeformed (solid line) and deformed (dashed line) meshes, for 4x4 finite elements, \(BC = 1\) and \(pe=[1, 9]\). In this figure, the element numbers are indicated at the center of the element, the nodemarks are circles, and the node \(N=21\), that is the left vertex on the top side of \(\omega\), in finite element 13, is the node with maximum displacement. The deformed mesh corresponds to the tangential displacement \(u_{tg}\) of the middle plane \(\omega\).

The increase of the displacements with the number of electrodes can also be observed in Tables 5 and 6, for each type of boundary condition. Nevertheless, we have obtained some numerical experiments where this phenomena is not verified, when more than 5 electrodes projections are considered. In fact, Figure 2 represents the objective function values of the Pareto optimal solutions, for the multi-objective problem (49), with 5x5 finite elements, \(m = 25\) and \(BC = 4\). These values increase with the number of electrodes projections, but the Pareto-optimal number of electrodes projections is 19 or 25 (to achieve a maximum node displacement it is enough to apply 19 or, even better, 25 electrodes). In fact, the graphic
depicted in Figure 2 confirms that for 5x5 finite elements, and the boundary condition $BC = 4$, there is no advantage in considering 20, 21, 22 or 23 electrodes instead of 19, since the actuator effect is better with only 19.

According to the results displayed in Tables 5 and 6, which refer to one up to five projections of the electrodes in the middle plane of the plate, we can delineate a summary of the best strategies for the location of the actuators. But before this summary, we first remark that the electric forces influence the mechanical displacements of the plate through the difference of the applied potentials on the top and bottom of the plate, and in the numerical experiments we have assumed $\phi_0 = 0$, all the actuators are placed on the upper face of the plate, ii) if $\phi_0 = -100V$ and $\phi_0 = 0$, all the actuators are placed on the upper face of the plate, ii) if $\phi_0 = 0$ and $\phi_0 = +100V$, all the actuators are placed on the lower face of the plate, iii) if $\phi_0 = a \neq 0$ and $\phi_0 = b \neq 0$, with $a + b = -100$, the number of actuators is the double of their orthogonal projections on $\omega$, and the electrodes are placed at the top and bottom of the plate. After this observation, we can now derive the following conclusions for the optimal electrode design, which induces the best actuator effect on the plate, under the influence of four different clamped boundary conditions:

- if $BC = 1$, the plate is clamped on the lateral side $bs \times (-h, h)$; place only one electrode either on the top or bottom of the plate, and at one extremity of this clamped side; place two up to five electrodes consecutively either on the top or bottom of the plate (for the possibilities either i) or ii), respectively, or alternatively place four up to ten consecutively on the top and bottom of the plate (for the possibility iii), such that their orthogonal projections on the middle plane $\omega$ are all aligned in a strip (with the width of one electrode and in the direction orthogonal to the bottom side $bs$ of $\omega$), and beginning with one projection placed either at the right or left extremity of $bs$,

- if $BC = 2$, the plate is clamped on the three lateral sides $ls \times (-h, h), bs \times (-h, h)$ and $rs \times (-h, h)$; place only one electrode either on the top or bottom of the plate and at one extremity of the free lateral side, which is $ts \times (-h, h)$; place two up to five electrodes consecutively either on the top or bottom of the plate (for the possibilities either i) or ii), respectively), or alternatively place four

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<th>$NFE$</th>
<th>$pe$</th>
<th>$3 \times 3$</th>
<th>$N - mv$</th>
<th>$pe$</th>
<th>$4 \times 4$</th>
<th>$N - mv$</th>
<th>$pe$</th>
<th>$5 \times 5$</th>
<th>$N - mv$</th>
</tr>
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</table>

Table 5: Solutions $pe$ and $N$ for $BC = 1, 2$
If \( BC = 3 \), the plate is clamped on the two consecutive lateral sides \( ls \times (-h, h) \) and \( rs \times (-h, h) \); place only one electrode either on the top or bottom of the plate, and at one extremity of these clamped lateral sides; place two up to five electrodes either on the top or bottom of the plate (for the possibilities either i) or ii), respectively), or alternatively place four up to ten consecutively on the top and bottom of the plate (for the possibility iii), such that their orthogonal projections begin with two juxtaposed projections in the middle of one free side \( ts \) of \( \omega \), and the remaining are aligned in a parallel strip with the direction of \( ts \) and towards the center of \( \omega \),

- if \( BC = 4 \), the plate is clamped on the two consecutive lateral sides \( bs \times (-h, h) \) and \( rs \times (-h, h) \); place two up to five electrodes consecutively either on the top or bottom of the plate (for the possibilities either i) or ii), respectively), or alternatively place four up to ten consecutively on the top and bottom of the plate (for the possibility iii), such that their orthogonal projections are aligned in a strip (with the width of one electrode) along one of the two free sides, either \( ls \) or \( bs \), and beginning

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</table>

Table 6: Solutions \( pe \) and \( N \) for \( BC = 3, 4 \)
Table 7: Optimal location of the projections of 3 electrodes, for the meshes 3x3, 4x4 and 5x5, and $BC = 2$

Figure 1: Undeformed and deformed meshes, for $BC = 1$, $pe=[1, 9]$ with one projection at the left upper corner of $\omega$.

Finally, and to conclude the numerical tests, we have considered nonzero mechanical transverse forces $f_{tv} = \int_{-\delta}^{\delta} f_3 \, dx_3 + g_3^+ + g_3^- = 100N$ (cf. (43)) and solved the right linear system in (52), whose unknown is the transverse displacement $u_{tv}$, for a 3x3 finite element mesh, $BC = 1$ and $pe=[1]$. Figure 3 represents the graphic of the corresponding discrete electric potential (35), as a function of the thickness variable $x_3$, for the finite element number 2, and Figure 4 exhibits the corresponding transverse displacement $u_{tv}$ of the middle plane $\omega$ of the plate.
5 Conclusion and future work

We have analyzed the actuator effect of the piezoelectric anisotropic plate model (16-19), as a function of the location of the applied electric potentials. The problem is formulated as an integer (single and multi-objective) optimization problem, strongly combinatorial, which has been successfully solved by genetic algorithms. In the numerical tests, a special case of anisotropy was considered, since the modified coefficients $p_{3\alpha\beta}$ and $p_{33}$, appearing in the model (16-19), and the reduced elastic coefficients $A_{\alpha\beta\gamma\rho}$, chosen in the numerical tests, have been assumed independent of the thickness variable $x_3$.

The asymptotic plate model used in this paper can be generalized to the case where the coefficients $p_{3\alpha\beta}$ and $p_{33}$ depend on the thickness variable $x_3$ (it corresponds to a generalization of Theorem 3.4 in Figueiredo and Leal [1]). In fact, if the coefficients $p_{3\alpha\beta}$ and $p_{33}$ depend on $x_3$, it can be proven that the limit electric potential $\varphi$ depends, not only, on the transverse displacement, (cf. (19)), but also, on the tangential displacement of the plate; consequently, the matrix $B$ defined in formula (22) is different (in particular it is non-symmetric) and therefore the linear system $Ku = F$ is more complex (now the tangential and transverse displacements are coupled). This more general asymptotic piezoelectric anisotropic plate model, also models a thin laminated piezoelectric plate made of two stacked layers of different piezoelectric materials - of course in this case the coefficients $p_{3\alpha\beta}$, $p_{33}$ and also $A_{\alpha\beta\gamma\rho}$ of the global laminated plate are functions of the thickness variable $x_3$. In a future work we intend to apply the same optimization procedure (that is, genetic algorithms), to study the actuator and sensor effect of this laminated piezoelectric plate. Moreover, in this future work, we will describe in detail all the features of the genetic algorithms applied to solve the problem.

References


Figure 3: Discrete electric potential at finite element 2: mesh 3x3, $BC = 1$, $pe = [1]$


Figure 4: Transverse displacement $u_{tz}$ of the middle plane $\omega$: mesh 3x3, BC = 1, pe=[1]
