Semidefinite lifts of polytopes

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with Richard Z. Robinson and Rekha Thomas (U.Washington)

Semidefinite Representations

A semidefinite representation of size k of a polytope P is a description

$$P = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists y \text{ s.t. } A_0 + \sum A_i x_i + \sum B_i y_i \succeq 0 \right\}$$

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This tells us how hard it is to optimize over P using semidefinite programming.

The Square

The $0/1$ square is the projection onto x_1 and x_2 of

$$\begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & x_1 & y \\ x_2 & y & x_2 \end{bmatrix} \succeq 0.$$

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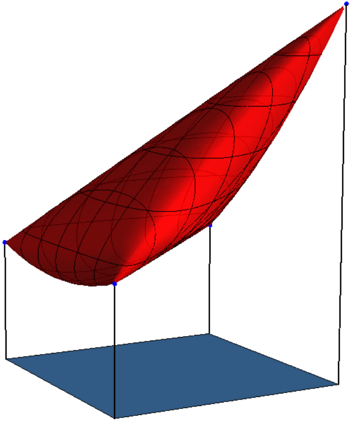

Definitions

Let P be a polytope with facets given by $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

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Semidefinite Factorizations

A PSD_k-factorization of M is a set of $k \times k$ positive semidefinite matrices $A_1, \cdots, A_m$ and $B_1, \cdots B_n$ such that $M_{i,j} = \langle A_i, B_j \rangle$.

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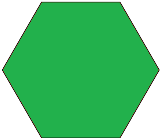
The psd rank of a polytope P is defined as

$$\mathrm{rank}_{psd}(P) := \mathrm{rank}_{psd}(S_P).$$

The Hexagon

Consider the regular hexagon.



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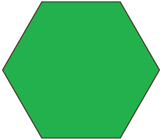

It has a 6×6 slack matrix.

$$\begin{bmatrix} 0 & 0 & 2 & 4 & 4 & 2 \\ 2 & 0 & 0 & 2 & 4 & 4 \\ 4 & 2 & 0 & 0 & 2 & 4 \\ 4 & 4 & 2 & 0 & 0 & 2 \\ 2 & 4 & 4 & 2 & 0 & 0 \\ 0 & 2 & 4 & 4 & 2 & 0 \end{bmatrix}$$

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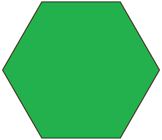

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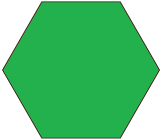

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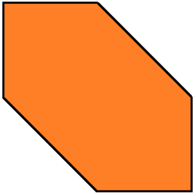

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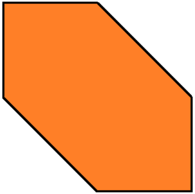

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$$H = \left\{ (x_1, x_2) : \begin{bmatrix} 1 & x_1 & x_2 & x_1 + x_2 \\ x_1 & 1 & y_1 & y_2 \\ x_2 & y_1 & 1 & y_3 \\ x_1 + x_2 & y_2 & y_3 & 1 \end{bmatrix} \succeq 0 \right\}$$

Bounds

Proposition (G.-Robinson-Thomas 2012)

All hexagons have psd rank 4, hence any m-gon has rank at most $4\lceil \frac{m}{6} \rceil$.

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Theorem (G.-Robinson-Thomas 2012)

Let P be a generic polytope with m vertices, then $\mathrm{rank}_{\mathrm{psd}}(P) \geq \sqrt[4]{m}$

Embarrassing state-of-art in $\mathbb{R}^2$

	min rank_{psd}	max rank_{psd}
	3	3
	3	3
	4	4
	4	4
	4 or 5	4 or 5
	4	4 or 5 or 6

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A polytope of dimension d does not have a semidefinite representation of size smaller than $d + 1$.

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Lemma

A polytope of dimension d does not have a semidefinite representation of size smaller than $d + 1$.

We want to make "small" $= d + 1$.

Characterization

Theorem (G.-Robinson-Thomas 2012)

Let P have dimension d. Then $\text{rank}_{\text{psd}}(P) = d + 1$ if and only if there exists an Hadamard square root matrix M of S_P such that $\text{rank}(M) = d + 1$.

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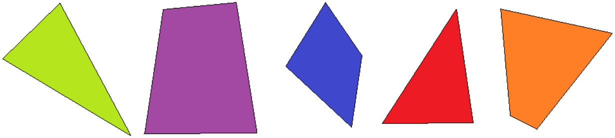
On the plane this is enough:

$\mathbb{R}^2$ characterization

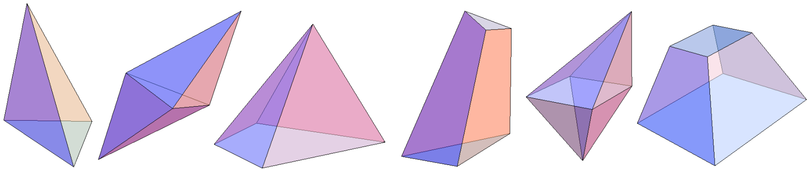
A 2-dimensional polytope is sdp-minimal iff it is a **triangle** or a **quadrilateral**.



A more interesting case

$\mathbb{R}^3$ characterization

A 3-dimensional polytope is sdp-minimal iff it is a **simplex**, a **bisimplex**, a **quadrilateral pyramid**, a **combinatorial triangular prism**, a **biplanar octahedra** or a **biplanar cuboid**.



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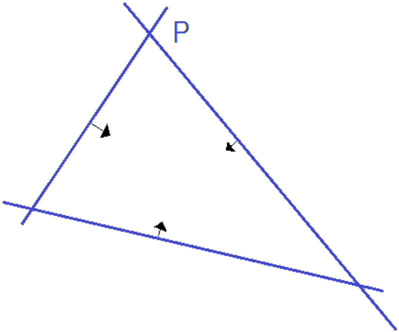

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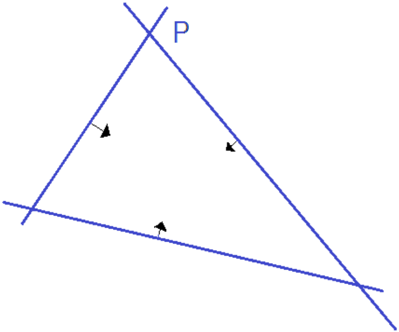

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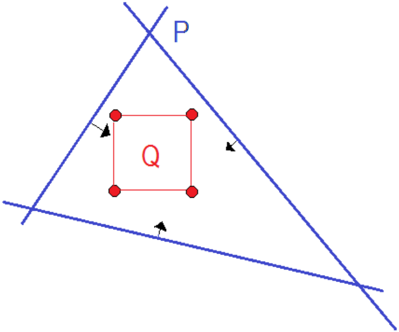

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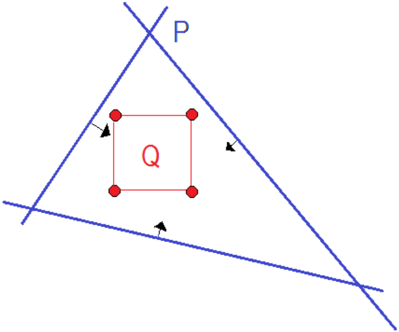

$$S_{P,Q} = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 7 & 9 & 4 & 3 \\ 5 & 1 & 5 & 9 \end{bmatrix}$$

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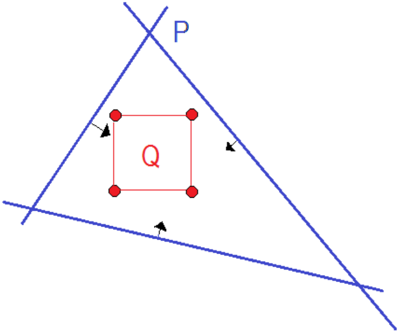

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All nonnegative matrices are of this type

How hard can it be? - Rank 3

Geometric Problem

Let $M = S_{P,Q}$ be a rank 3 nonnegative matrix. $\mathrm{rank}_{\mathrm{psd}}(M) = 2$ if and only if we can fit a (half)-conic between Q and P.

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Example:

$$M_\varepsilon = S_{C,(1-\varepsilon)C} = \begin{bmatrix} 2-\varepsilon & 2-\varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 2-\varepsilon & 2-\varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 2-\varepsilon & 2-\varepsilon \\ 2-\varepsilon & \varepsilon & \varepsilon & 2-\varepsilon \end{bmatrix}$$

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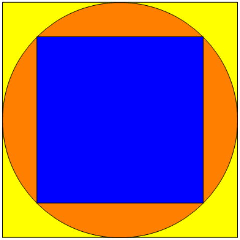
$$\text{rank}_{\text{psd}} M_\varepsilon = \begin{cases} 1 & \text{if } \varepsilon = 1; \\ 2 & \text{if } \varepsilon \in [1-\sqrt{2}/2, 1); \\ 3 & \text{if } \varepsilon \in [0, 1-\sqrt{2}/2). \end{cases}$$



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Theorem - G.-Robinson-Thomas 2013

MIN PSD RANK can be solved in time $(pq)^{O(d^{2.5})}$ for $M \in \mathbb{R}_+^{p \times q}$ and $\text{rank}(M) = d = \binom{k+1}{2}$.

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In particular, **for fixed rank, MIN PSD RANK can be solved in polynomial time.**

Conclusion

PSD Factorization/rank is an exciting area of research with many recent breakthroughs and many open questions.

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To read more on this:

Worst-case Results for Positive Semidefinite Rank - G., Robinson and Thomas - arXiv:1305.4600

Polytopes of Minimum Positive Semidefinite Rank - G., Robinson and Thomas - arXiv:1205.5306

Lifts of convex sets and cone factorizations - G. , Parrilo and Thomas - Math of OR

Thank you