Sums of Squares on the Hypercube

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Section 1

Introduction

Nonnegativity of a polynomial

Let $I \subseteq \mathbb{R}[x]$ be an ideal:

$$\mathcal{P}(I) = \{p \in \mathbb{R}[I] \; : \; p \text{ is nonnegative on } \mathcal{V}_{\mathbb{R}}(I)\}\,.$$

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A typical strategy is to approximate $\mathcal{P}(I)$ by

$$\Sigma(I) = \left\{p \in \mathbb{R}[I] \ : \ p \equiv \sum_{i=1}^{t} h_i^2 \text{ for some } h_i \in \mathbb{R}[I]\right\},$$

and its truncations

$$\Sigma_k(I) = \left\{p \in \mathbb{R}[I] \ : \ p \equiv \sum_{i=1}^{t} h_i^2 \text{ for some } h_i \in \mathbb{R}_k[I]\right\}.$$

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When are sums of squares enough?

Theorem (Hilbert 1888)

$\Sigma_k(\mathbb{R}^n) = \mathcal{P}_{2k}(\mathbb{R}^n)$ *if and only if* $n = 1$, $k = 1$ *or* $(n, k) = (2, 2)$.

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Theorem (Scheiderer 1999)

If $\dim(\mathcal{V}_{\mathbb{R}}(I)) \geq 3$ *then* $\Sigma(I) \neq \mathcal{P}(I)$.

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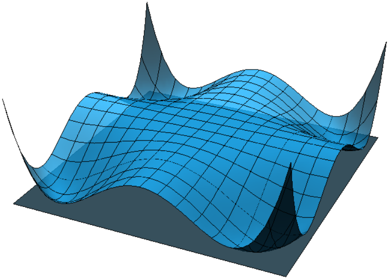

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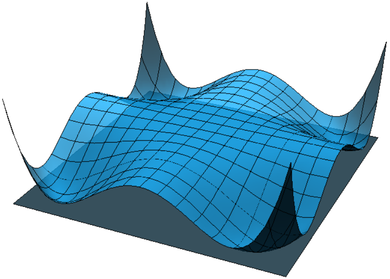

$$M(x,y) = (x^2+y^2+1)\left(\frac{x^3y+xy^3-2xy}{x^2+y^2}\right)^2 + \left(\frac{x^2-y^2}{x^2+y^2}\right)^2.$$

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In other words, we want to bound the degrees of the denominators in the rational functions used.

Advantages and Disadavantages

Schmudgen's Positivstellensatz

If $\mathcal{V}_{\mathbb{R}}(I)$ is compact, p positive on $\mathcal{V}_{\mathbb{R}}(I)$ implies p is k-sos for some k.

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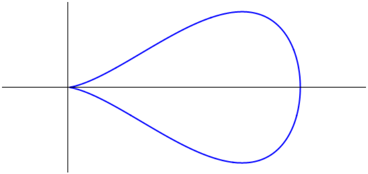
- Optimizing over the set of all k-rsos polynomials is not as easy.

Example

Consider the teardrop curve given by $\mathcal{V}_{\mathbb{R}}(\langle x^4 - x^3 + y^2 \rangle)$.



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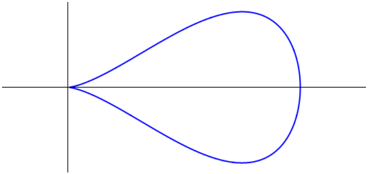

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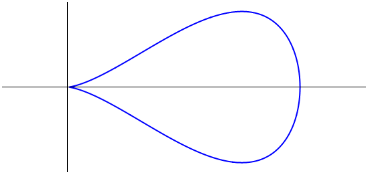

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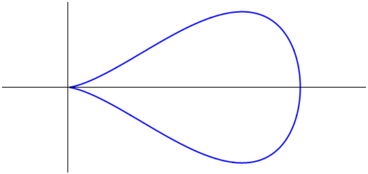

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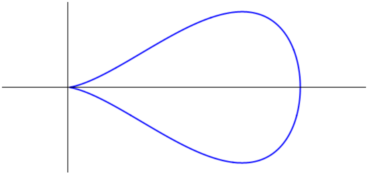

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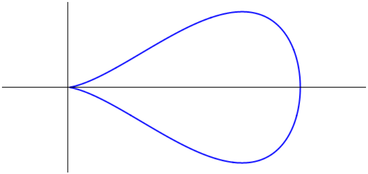

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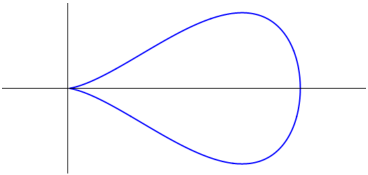

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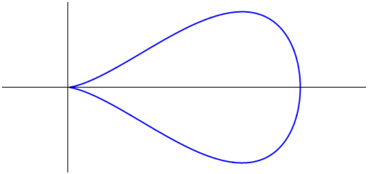

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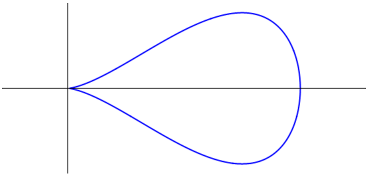

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Multipliers make the certificates less sensitive to singularities.

Section 2

Upper bounds on multipliers

Finite Varieties

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What bounds can we give for rsos polynomials?

Lemma

Let $\ell : \mathbb{R}[X]_{2d} \to \mathbb{R}$ be given by $\ell(f) = \sum_{v \in X} \mu_v f(v)$ with all $\mu_v \neq 0$. Suppose that ℓ is nonnegative on $\Sigma_d(X)$. Then

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Upper Bound Theorem

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With this lemma we can prove our main upper bound theorem.

Theorem

Let $p \in \mathbb{R}[I]_{2s}$ be nonnegative on X. Suppose that for some $k \in \mathbb{N}$ we have

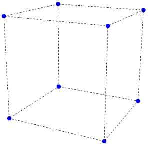
$$H_X(k+s) + H_X(k) > H_X(2k+2s).$$

Then p is $(k+s)$-rsos on X, i.e. there exists $h \in \Sigma_k(X)$ such that $ph \in \Sigma_{s+k}(X)$.

The n-cube

We are interested in the n-cube:

$$C_n = \{0,1\}^n = \{x \in \mathbb{R}^n \ : \ x_i^2 - x_i = 0,\ i = 1, \cdots, n\} = \mathcal{V}(I_n).$$



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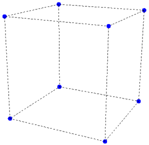

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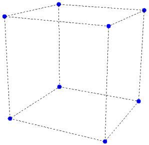

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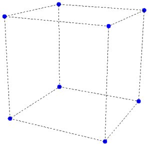

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Corollary

Every nonnegative quadratic polynomial on C_n is $(\lfloor n/2 \rfloor + 1)$-rsos.

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Open Question: Is the increased degree needed?

Section 3

Lower bounds on hypercube multipliers

Hypercube

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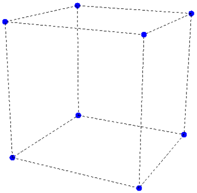

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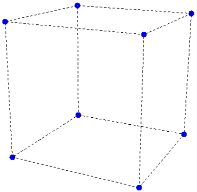

Cube C_3

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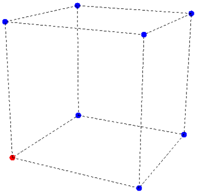

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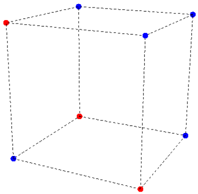

Level T_1

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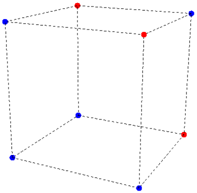

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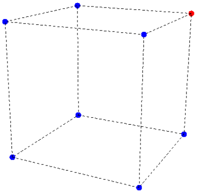

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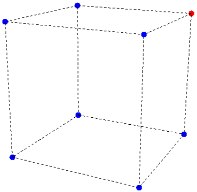

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Symmetric polynomials appear naturally in combinatorial optimization, and we want lower bounds for the degree of nonnegativity certificates.

Symmetric group representations

Symmetric group representations

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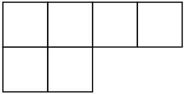

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Irreducible S_n-modules are precisely given by the Specht modules S^λ.

S_n action on $\mathbb{R}[I]$:

The action of S_n in $\mathbb{R}[I]_k$, for $k \leq \lfloor n/2 \rfloor$ decomposes as follows:

$$\mathbb{R}[I]_k = \mathbb{R}[I]_{=0} \oplus \mathbb{R}[I]_{=1} \oplus \mathbb{R}[I]_{=2} \oplus \cdots \oplus \mathbb{R}[I]_{=k}$$

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& S^{[n,0]} & & S^{[n,0]} & & S^{[n,0]} & & \dots & & S^{[n,0]} \\
& & & \oplus & & \oplus & & & & \oplus \\
& & & S^{[n-1,1]} & & S^{[n-1,1]} & & \dots & & S^{[n-1,1]} \\
& & & & & \oplus & & & & \oplus \\
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\end{array}
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Let M_j be the first copy of $S^{[n-j,j]}$ to appear, then

$$\mathbb{R}[I]_k = \bigoplus_{j=0}^{k} M_j \oplus (k - \sum x_i) M_j \oplus \cdots \oplus (k - \sum x_i)^{k-j} M_j .$$

An explicit decomposition

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Polynomials in M_j do not vanish in T_k for $j \leq k \geq n - j$. This is enough for our main lemma

Lemma

Suppose $f \in \mathbb{R}_d[I_n]$, vanishes on T_t. If $d \leq t \leq n - d$, then f is properly divisible by $\ell = t - \sum x_i$.

The bound

Theorem

Suppose $f \in \mathbb{R}_t[I_n]$ with $t \leq n/2$ is an S_n-invariant polynomial and f is properly divisible by $\ell = t - (x_1 + \cdots + x_n)$ to odd order. Then f is not d-rsos for $d \leq t$.

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In particular:

Theorem

Let $k = \lfloor \frac{n}{2} \rfloor$ and let $f \in \mathbb{R}[I_n]$ be given by

$$f = (x_1 + \cdots + x_n - k)(x_1 + \cdots + x_n - k - 1).$$

Then f is nonnegative on C_n but f is not k-rsos.

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This shows our upper bound was tight.

Section 4

Applications

Globally nonnegative polynomials

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Let $k = \lfloor \frac{n}{2} \rfloor$. There exists a polynomial p of degree 4 nonnegative on $\mathbb{R}^n$ which is not k-rsos in $\mathbb{R}[x_1, \ldots, x_n]$.

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MaxCut

The maxcut problem over K_n can be reduced to

Binary polynomial formulation of MaxCut

$$\max p(x) = \sum_{i \neq j} (1 - x_i) x_j \text{ s.t. } x \in C_n$$

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Note that p attains its maximum in C_n at T_k and T_{k+1} so

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A weaker version can now be proved.

Theorem

If $n = 2k + 1$, $(p_\omega)_{\max} = (p_\omega)^{k+1}_{\text{rsos}}$ for all weights or $(p_\omega)^{k+2}_{\text{rsos}}$ if we want positive multipliers.

Thank You