Lifts of convex sets and cone factorizations

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with Pablo Parrilo (MIT) and Rekha Thomas (U.Washington)

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For every even set $A \subseteq \{1, \ldots, n\}$,

$$\sum_{i \in A} x_i - \sum_{i \notin A} x_i \leq |A| - 1$$

is a facet, so we have at least 2^{n-1} facets.

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$O(n^2)$ variables and $O(n^2)$ constraints.

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Canonical LP Lift

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The smallest k such that P has a $\mathbb{R}^k_+$-lift is a much better measure of "LP-complexity" than number of facets and vertices.

Two definitions

Let P be a polytope with facets defined by $h_1(\mathbf{x}) \geq 0, \ldots, h_f(\mathbf{x}) \geq 0$, and vertices $p_1, \ldots, p_v$.

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Nonnegative Factorization

Given a nonnegative matrix $M \in \mathbb{R}_+^{n \times m}$ we say that it has a k-nonnegative factorization, or a $\mathbb{R}_+^k$-factorization if there exist matrices $A \in \mathbb{R}_+^{n \times k}$ and $B \in \mathbb{R}_+^{k \times m}$ such that

$$M = A \cdot B.$$

Yannakakis' Theorem

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- Can we include symmetry in the result?

The Hexagon

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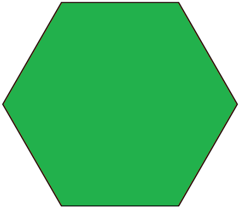

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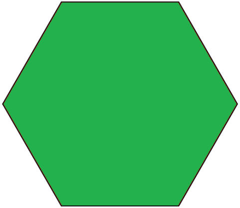

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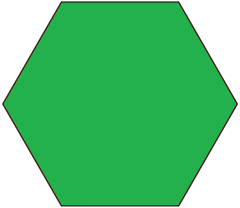

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$$\begin{bmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$

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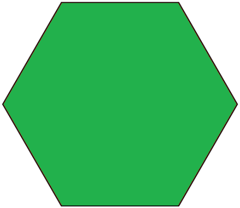

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Hexagon - continued

It is the projection of the slice of $\mathbb{R}^5_+$ cut out by

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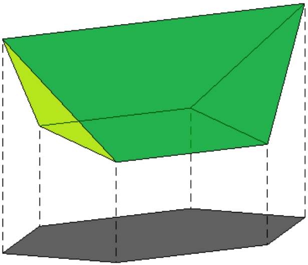

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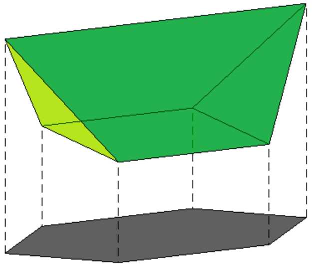

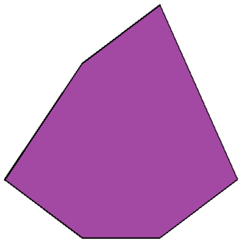

For irregular hexagons a $\mathbb{R}^6_+$-lift is the only we can have.

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We also need to generalize the nonnegative factorizations.

K-factorizations

Recall that if $K \subseteq \mathbb{R}^l$ is a closed convex cone, $K^* \subseteq \mathbb{R}^l$ is its dual cone, defined by

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We can now generalize Yannakakis.

Theorem (G-Parrilo-Thomas)

A polytope P has a K-lift if and only if S_P has a K-factorization.

The Square

The $0/1$ square is the projection onto x and y of the slice of PSD_3 given by

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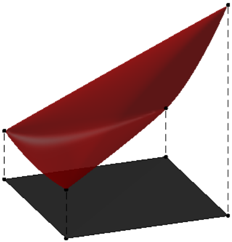

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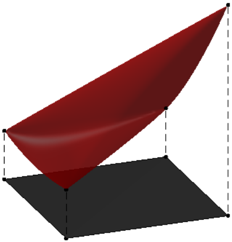

Its slack matrix is given by

$$S_P = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

and should factorize in PSD_3.

Square - continued

$$S_P = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

is factorized by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

for the rows and

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Note that this generalizes the slack matrix.

Generalized Yannakakis for convex sets

We can then define a K-factorization of S_C as a pair of maps

$$A : \operatorname{ext}(C) \to K \qquad B : \operatorname{ext}(C^\circ) \to K^*$$

such that

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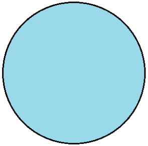
A convex set C has a K-lift if and only if S_C has a K-factorization.

The Disk

The unit disk D is the projection onto x and y of the slice of PSD_2 given by

$$\begin{bmatrix} 1+x & y \\ y & 1-x \end{bmatrix} \succeq 0.$$



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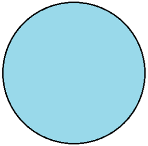

$D^\circ = D$, there must be $A : S^1 \to \mathrm{PSD}_2$ and $B : S^1 \to \mathrm{PSD}_2$ such that $\langle A(x), B(y) \rangle = 1 - \langle x, y \rangle$

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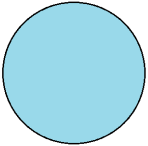

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$$A(x,y) = \begin{bmatrix} 1+x & y \\ y & 1-x \end{bmatrix}, \quad B(x,y) = \begin{bmatrix} 1-x & -y \\ -y & 1+x \end{bmatrix}.$$

Cone ranks of polytopes

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We are specially interested in $\text{rank}_{\text{psd}}(M)$.

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Proposition

If $M \in \mathbb{R}_+^{n \times n}$ is zero on the diagonal and positive everywhere else then $\text{rank}_+(M) \geq k$, where k is the smallest integer such that $n \leq \binom{k}{\lfloor k/2 \rfloor}$.

Bounds for matrices - 2

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If $M \in \mathbb{R}^{p \times q}$ has rank k, then the matrix M' obtained by squaring each entry of M has psd-rank at most k.

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rank_{+} can be arbitrarily larger than rank and $\mathrm{rank}_{\mathrm{psd}}$.

Bounds for polytopes - LP

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- *If p is the size of the largest antichain in $L(P)$, then $\text{rank}_+(P) \leq k$ where k is the smallest integer such that $p \leq \binom{k}{\lfloor k/2 \rfloor}$.*

Bounds for polytopes - LP

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- $n_P = 28 \Rightarrow \text{rank}_+(P) \geq \log_2(28) \approx 4.807$.
- $n_{\text{edges}} = 12$, $\binom{5}{2} = 10$, $\binom{6}{3} = 20$, hence $\text{rank}_+(P) \geq 6$.

Bounds for polytopes - SDP

Theorem

If a polytope P in $\mathbb{R}^n$ has $\text{rank}_{psd} = k$ than it has at most $k^{O(k^2 n)}$ facets.

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Open questions:

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Open questions:

- Can we find a separation between rank_{psd} and rank_+ for polytopes?
- Recently, [Fiorini-Massar-Pokutta-Tiwary-de Wolf] proved $\text{rank}_+(\text{TSP})$ grows exponentially. What about rank_{psd}?

Symmetric Lifts

In the LP case there has been much interest in symmetric lifts. [Kaibel-Pashkovich-Theis]

Symmetric lifts

Let P be a polytope and $P = \Phi(K \cap L)$ a lift of P. We say the lift is **symmetric** if there exists a group homomorphism sending $g \in \mathrm{Aut}(P)$ to $\psi_g \in \mathrm{Aut}(K)$ such that $\psi_g(L) = L$ and $\Phi \circ \psi_g = g$.

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Symmetric lift preserves symmetries of the lifted objects.

Common lift-and-project methods are symmetric (w.r.t. permutation of variables): LS, SA, Las...

Example:The square

Recall the lift of the $0/1$ square

$$\begin{bmatrix} 1 & x & y \\ x & x & z \\ y & z & y \end{bmatrix} \succeq 0.$$

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$$\phi_g(A) = \begin{bmatrix} 1 & y & x \\ y & y & z \\ x & z & x \end{bmatrix} = P_{23}AP_{23},$$

$$\phi_h(A) = \begin{bmatrix} 1 & 1-x & y \\ 1-x & 1-x & y-z \\ y & y-z & y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

generate a homomorphism so the lift is symmetric.

Example 2 : Regular *n*-gons

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Note that we know that there are actually $O(\log(n))$ dimensional lifts of these polytopes [Ben-Tal, Nemirovski].

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For p prime the smallest k for which there exists a symmetric $\mathbb{R}^k_+$-lift of the p-gon is p.

Note that we know that there are actually $O(\log(n))$ dimensional lifts of these polytopes [Ben-Tal, Nemirovski].

Open (small) problem: prove that the smallest symmetric lift of an n-gon is to $\mathbb{R}^n_+$.

Symmetric Yannakakis

K-Factorization

Given a polytope P and its slack matrix $S_P \in \mathbb{R}_+^{n\times m}$ and its K-factorization given by $a_1, \dots a_n \in K$, $b_1, \dots, b_m \in K^*$, we say that it is symmetric if there is an homomorphism $\phi : \mathrm{Aut}(P) \to \mathrm{Aut}(K)$ such that if g send the i-th vertex to the j-th vertex, $\phi(a_i) = a_j$.

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Theorem (G-Parrilo-Thomas)

A convex set C has a symmetric K-lift if and only if S_C has a symmetric K-factorization.

Matchings

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Given the complete graph $K_n = ([n], E_n)$ a **matching** is a collection M of edges such that there's one and only one edge incident to each vertex.

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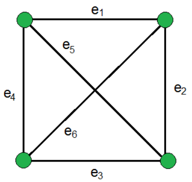

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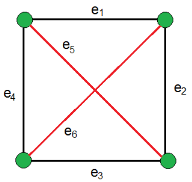

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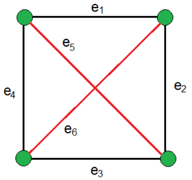

$\chi_M \in \{0, 1\}^{E_n}$ is the indicator vector of M.
For this example $\chi_M = (0, 0, 0, 0, 1, 1)$.

The matching Polytope

MaxMatch

Given a complete graph K_{2n} with edge weights $\omega : E_n \to \mathbb{R}$, find the matching with maximum weight.

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This polytope is the Matching Polytope, denoted PMatch_{2n}.

Symmetric lifts of matching polytope

Yannakakis

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What about non-symmetric?

With other versions of the matching polytope, Kaibel, Pashkovich and Theis show that symmetry does matter, but the general question is still open.

Symmetry in SDP

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The end

Thank You