Sums of Squares in Combinatorial optimization

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CMUC - Universidade de Coimbra

17th March 2012 - *2nd Combinatorics Day* - Coimbra

Stable Sets

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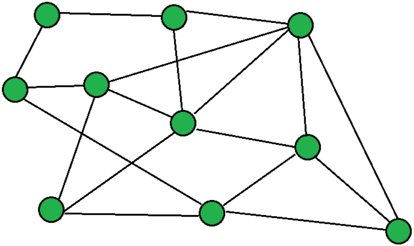

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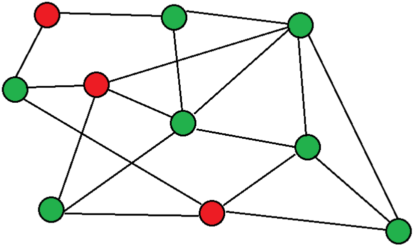

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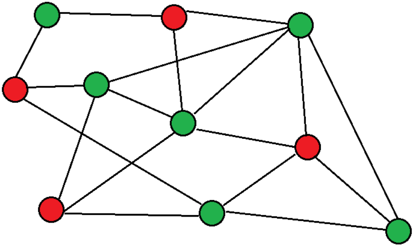

The Stable Set Problem

The stable set problem for G, given some vertex weights ω is:

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- **this problem is NP-hard in general.**

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- the polytope $\mathrm{STAB}(G)$ is then defined as the convex hull of the vectors in S_G.

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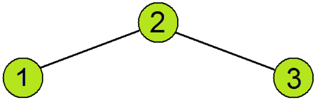

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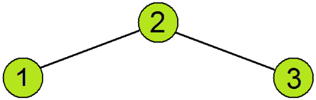

$$S_G = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,0,1)\}$$

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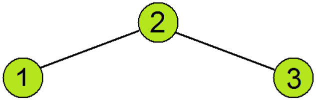

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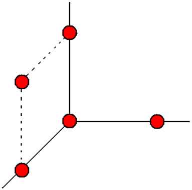

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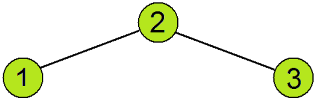

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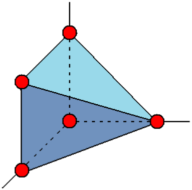

Reformulation of the Problem

Stable Set Problem Reformulated

Given a graph $G = ([n], E)$ and a weight vector $\omega \in \mathbb{R}^n$, solve

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However, finding $\mathrm{STAB}(G)$ is as hard as solving the original problem.

We intend to find approximations for it.

Fractional Stable Set Polytope

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It is in general not a very good relaxation.

Definition of Theta Body

Definition (Lovász $\sim$ 1980)

Given a graph $G = ([n], E)$ we define its theta body, $\mathrm{TH}(G)$, as the set of all vectors $x \in \mathbb{R}^n$ such that

$$\begin{bmatrix} 1 & x^t \\ x & U \end{bmatrix} \succeq 0$$

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Theorem (Lovász $\sim$ 1980)

The relaxation is tight, i.e. $TH(G) = STAB(G)$, if and only if the graph G is perfect.

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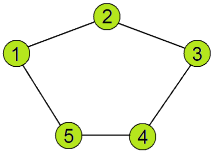

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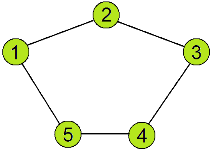

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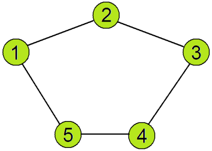

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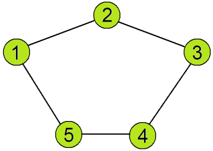

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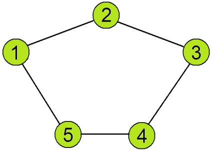

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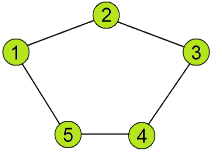

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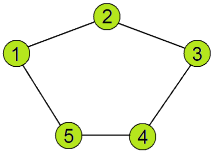

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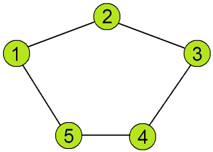

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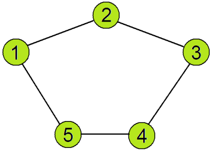

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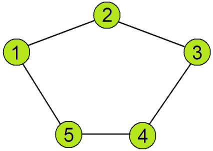

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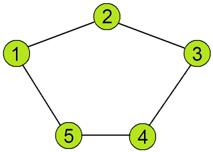

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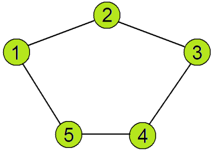

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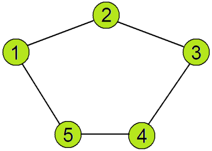

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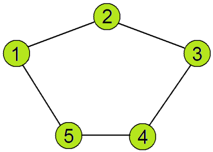

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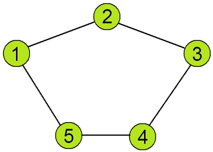

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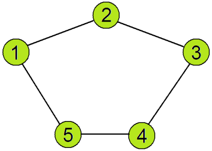

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In this case $\mathrm{TH}(G) \neq \mathrm{STAB}(G)$

k-Sums of Squares

Let $I \subseteq \mathbb{R}[x]$ be an ideal.

$f \in \mathbb{R}[x]$ is k-sos modulo I if and only if

$$f \equiv ({h_1}^2 + {h_2}^2 + \ldots + {h_m}^2) \mod I,$$

for some polynomials $h_1, \ldots, h_m$ with degree less or equal k.

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For any p in $\mathcal{V}_{\mathbb{R}}(I)$ we have

$$f(p) = {h_1}^2(p) + ... + {h_m}^2(p) \geq 0,$$

so being k-sos modulo I, implies being nonnegative on $\mathcal{V}_{\mathbb{R}}(I)$.

Convex Hulls of Varieties

We want to use this tool to approximate $S = \mathcal{V}_{\mathbb{R}}(I)$. Note that

Convex Hull

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We have $\overline{\text{conv}(S)} \subseteq \cdots \subseteq \text{TH}_3(I) \subseteq \text{TH}_2(I) \subseteq \text{TH}_1(I)$.

Theta body - Example

(Loading...)

$\text{TH}_2(I)$ for $I = \langle x(x^2 + y^2) - x^4 - x^2y^2 - y^4 \rangle$.

Back to Stable Sets

For a graph G, let $S_G = \{\chi_S \ : \ S \text{ is stable}\}$ and $I_G = \mathcal{I}(S_G)$, then $\text{TH}_k(I_G)$ is a hierarchy approximating $\text{STAB}(G)$.

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$$I_G = \left\langle x_1^2 - x_1, x_2^2 - x_2, \cdots, x_n^2 - x_n, x_i x_j \mid \text{ for all } \{i,j\} \in E \right\rangle .$$

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This relates the new relaxations to the Lovász theta body.

Theorem

For any graph G, $\mathrm{TH}(G) = \mathrm{TH}_1(I_G)$*.*

Certificates

If $H \subseteq G$ is a clique, then

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Since, modulo I_G, $x_i^2 = x_i$ and $x_i x_j = 0$ for $\{i, j\} \in E$,

$$1 - \sum_{i \in H} x_i \equiv \left(1 - \sum_{i \in H} x_i\right)^2 \text{ modulo } I_G$$

hence it is 1-sos and valid on $\mathrm{TH}_1(I_G)$.

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$\mathrm{TH}_2(I_G) = \mathrm{STAB}(G)$ for h-perfect graphs.

Further Thoughts on Stable Sets

Since G is TH_1-exact if and only if it is perfect, makes sense to ask

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We know for example that odd cycle and odd wheel inequalities are captured in $\text{TH}_2(I_G)$. Little else has been done, which raises another interesting open question.

Question

Find an explicit family G_n for which $\text{TH}_n(I_{G_n}) \neq \text{STAB}(G_n)$ for all n.

Combinatorial Moment Matrices

How to optimize over these bodies?

Combinatorial Moment Matrices

How to optimize over these bodies? The moment approach.

Let

$$\mathcal{B} = \{1 = f_0, x_1 = f_1, ..., x_n = f_n, f_{n+1}, ...\}$$

be a basis of $\mathbb{R}[x]/I$ and $\mathcal{B}_k$ its truncation at degree k.

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Consider the polynomial vector $f^k(x) = (f_i(x))_{f_i \in \mathcal{B}_k}$ then

$$(f^k(x))(f^k(x))^t = \sum_{f_i \in \mathcal{B}} A_i f_i(x)$$

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for some symmetric matrices A_i. Given a vector y indexed by the elements in $\mathcal{B}$ we define the ***k*-th truncated combinatorial moment matrix** of y as

$$M_{\mathcal{B},k}(y) = \sum_{f_i \in \mathcal{B}} A_i y_{f_i}.$$

Combinatorial Moment Matrices - Example

Let $I = \langle x_1^2 - x_1, x_2^2 - x_2, x_3^2 - x_3 \rangle \subset \mathbb{R}[x_1, x_2, x_3]$,

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$$\begin{array}{c} \\ 1 \\ x_1 \\ x_2 \\ x_3 \\ x_1x_2 \\ x_1x_3 \\ x_2x_3 \\ x_1x_2x_3 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & x_1 & x_2 & x_3 & x_1x_2 & x_1x_3 & x_2x_3 & x_1x_2x_3 \end{array} \\ \begin{bmatrix} \mathbf{y_0} & \mathbf{y_1} & \mathbf{y_2} & \mathbf{y_3} & y_{12} & y_{13} & y_{23} & y_{123} \\ \mathbf{y_1} & \mathbf{y_1} & \mathbf{y_{12}} & \mathbf{y_{13}} & y_{12} & y_{13} & y_{123} & y_{123} \\ \mathbf{y_2} & \mathbf{y_{12}} & \mathbf{y_2} & \mathbf{y_{23}} & y_{12} & y_{123} & y_{23} & y_{123} \\ \mathbf{y_3} & \mathbf{y_{13}} & \mathbf{y_{23}} & \mathbf{y_3} & y_{123} & y_{13} & y_{23} & y_{123} \\ y_{12} & y_{12} & y_{12} & y_{123} & y_{12} & y_{123} & y_{123} & y_{123} \\ y_{13} & y_{13} & y_{123} & y_{13} & y_{123} & y_{13} & y_{123} & y_{123} \\ y_{23} & y_{123} & y_{23} & y_{23} & y_{123} & y_{123} & y_{23} & y_{123} \\ y_{123} & y_{123} & y_{123} & y_{123} & y_{123} & y_{123} & y_{123} & y_{123} \end{bmatrix} \end{array}$$

Combinatorial Moment Matrices - Example

Let $I = \langle x_1^2 - x_1, x_2^2 - x_2, x_3^2 - x_3 \rangle \subset \mathbb{R}[x_1, x_2, x_3]$, pick

$$\begin{array}{cccccccccc} \mathcal{B} = \{ & 1, & x_1, & x_2, & x_3, & x_1x_2, & x_1x_3, & x_2x_3, & x_1x_2x_3 & \} \\ y = (& y_0, & y_1, & y_2, & y_3, & y_{12}, & y_{13}, & y_{23}, & y_{123} &). \end{array}$$

$M_{\mathcal{B},2}(y)$ is given by:

$$\begin{array}{c|cccccccc} & 1 & x_1 & x_2 & x_3 & x_1x_2 & x_1x_3 & x_2x_3 & x_1x_2x_3 \\ \hline 1 & \mathbf{y_0} & \mathbf{y_1} & \mathbf{y_2} & \mathbf{y_3} & \mathbf{y_{12}} & \mathbf{y_{13}} & \mathbf{y_{23}} & y_{123} \\ x_1 & \mathbf{y_1} & \mathbf{y_1} & \mathbf{y_{12}} & \mathbf{y_{13}} & \mathbf{y_{12}} & \mathbf{y_{13}} & \mathbf{y_{123}} & y_{123} \\ x_2 & \mathbf{y_2} & \mathbf{y_{12}} & \mathbf{y_2} & \mathbf{y_{23}} & \mathbf{y_{12}} & \mathbf{y_{123}} & \mathbf{y_{23}} & y_{123} \\ x_3 & \mathbf{y_3} & \mathbf{y_{13}} & \mathbf{y_{23}} & \mathbf{y_3} & \mathbf{y_{123}} & \mathbf{y_{13}} & \mathbf{y_{23}} & y_{123} \\ x_1x_2 & \mathbf{y_{12}} & \mathbf{y_{12}} & \mathbf{y_{12}} & \mathbf{y_{123}} & \mathbf{y_{12}} & \mathbf{y_{123}} & \mathbf{y_{123}} & y_{123} \\ x_1x_3 & \mathbf{y_{13}} & \mathbf{y_{13}} & \mathbf{y_{123}} & \mathbf{y_{13}} & \mathbf{y_{123}} & \mathbf{y_{13}} & \mathbf{y_{123}} & y_{123} \\ x_2x_3 & \mathbf{y_{23}} & \mathbf{y_{123}} & \mathbf{y_{23}} & \mathbf{y_{23}} & \mathbf{y_{123}} & \mathbf{y_{123}} & \mathbf{y_{23}} & y_{123} \\ x_1x_2x_3 & y_{123} & y_{123} & y_{123} & y_{123} & y_{123} & y_{123} & y_{123} & y_{123} \end{array}$$

Moment relaxation

Define the convex body

$$Q_k(I) = \{y \in \mathbb{R}^{\mathcal{B}} : y_0 = 1, M_{\mathcal{B},k}(y) \succeq 0\}.$$

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Theorem

For any ideal I, $\overline{L_k(I)} = \mathrm{TH}_k(I)$.

This allows us to optimize over $\mathrm{TH}_k(I)$ efficiently.

Convergence of Theta Bodies

Theorem

If $\mathcal{V}_{\mathbb{R}}(I)$ is finite the hierarchy converges in finitely many steps.

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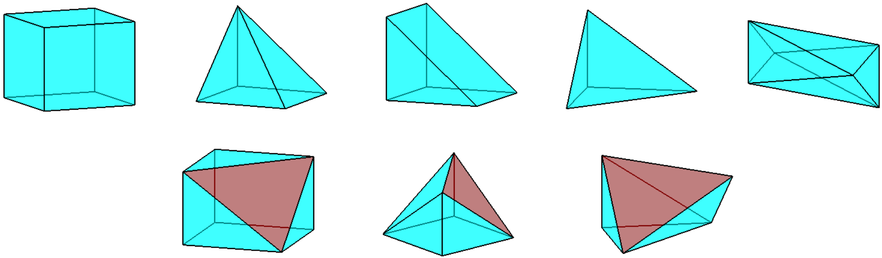
Theorem

If $S \subseteq \mathbb{R}^n$ is finite and $I = \mathcal{I}(S)$ then $\mathrm{TH}_1(I) = \mathrm{conv}(S)$ if and only if S is the set of vertices of a 2-level polytope.

Examples in $\mathbb{R}^3$



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Definition

Given a graph $G = (V, E)$ and a partition V_1, V_2 of V the set C of edges between V_1 and V_2 is called a **cut**.

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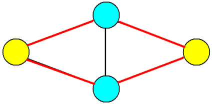

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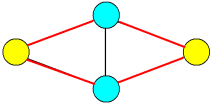

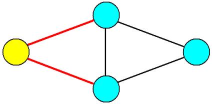

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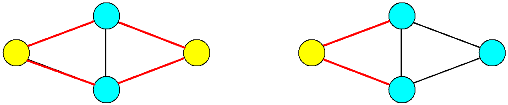

The Problem

Given edge weights α we want to find which cut C maximizes

$$\alpha(C) := \sum_{e \in C} \alpha_e.$$

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Definition

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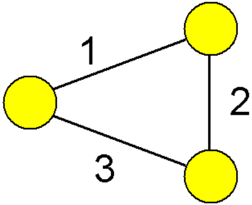

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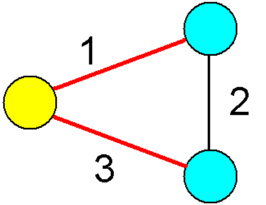

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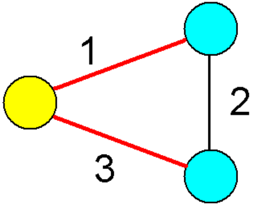

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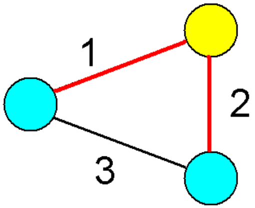

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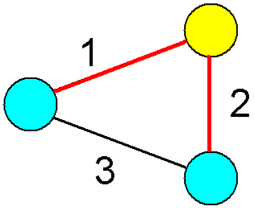

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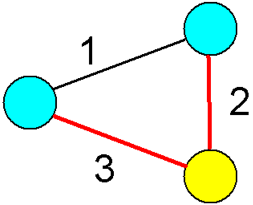

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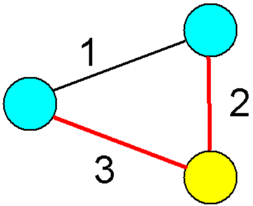

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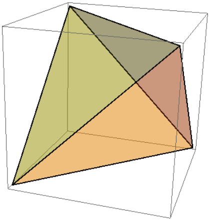

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LP formulation and relaxation

Our plan is to consider the vanishing ideal

$$I_G := \{f \in \mathbb{R}[\mathbf{x}] : f(\chi_C) = 0 \text{ for all cuts of } G\},$$

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This again can be done 'efficiently' using combinatorial moment matrices.

T-Joins

Let $G = (V, E)$ be a graph. Given an even set $T \subseteq V$ we define a T**-join** to be a subgraph of G with odd degree precisely in the vertices of T.

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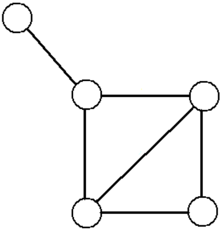

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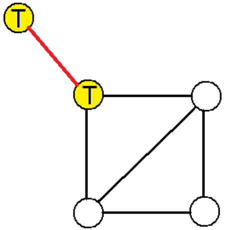

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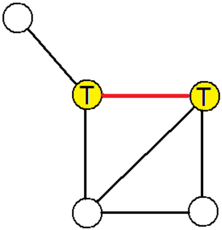

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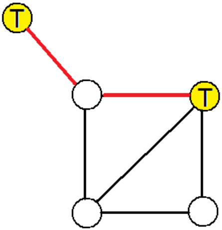

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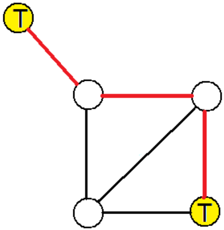

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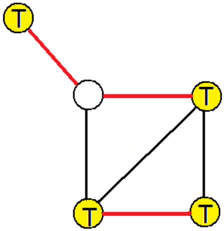

The Ideal

Theorem

If G is connected then the set

$$\{x_e^2 - 1 : \; e \in E)\} \cup \{1 - \mathbf{x}^A : \; A \subseteq E, A \text{ circuit in } G\}$$

generates I_G, and

$$\mathcal{B} := \{\mathbf{x}^{F_T} \; : \; T \subseteq [n], \, |T| \, \text{even}\}$$

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$\mathcal{B}_k$ with the set of all even $T \subseteq V$ such that $d_T \leq k$.

General Cut Theta body

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The set $TH_k(I_G)$ is given by

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The First Cut Theta Body

Cut Theta Body

Given a graph $G = (V, E)$ the body $TH_1(I_G)$ is the set of all $x \in \mathbb{R}^E$ such that

$$\begin{bmatrix} 1 & x^t \\ x & U \end{bmatrix} \succeq 0$$

for some a symmetric $U \in \mathbb{R}^{E \times E}$ s.t.

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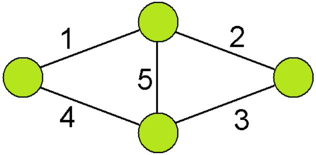
Example



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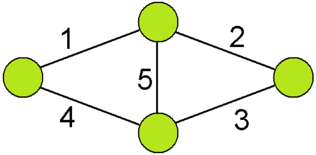

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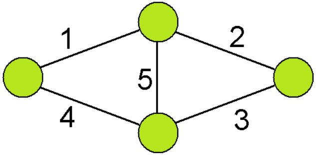

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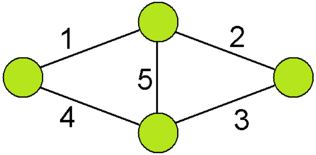

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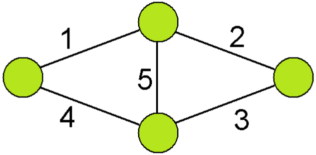

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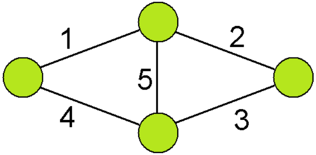

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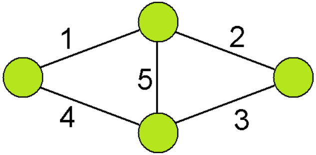

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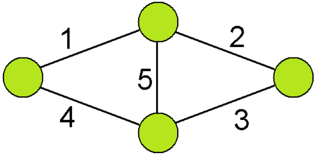

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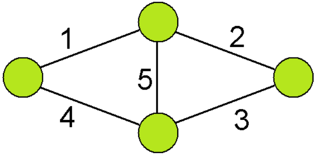

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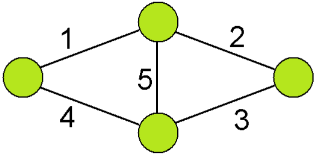

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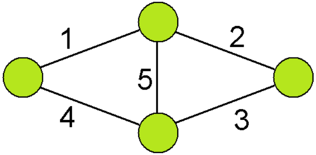

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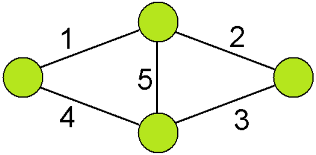

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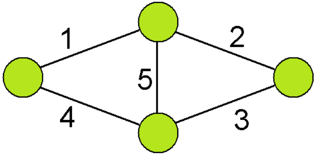

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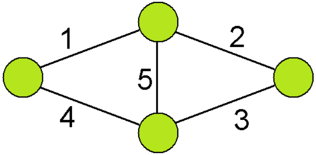

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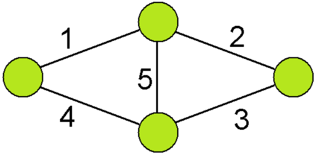

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Analogue to the stable set problem, it makes sense to have the following definition:

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Theorem

A graph is cut-perfect if and only if it has no K_5 minor and no chordless cycle of size larger than 4.

Some remarks

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- How general are these relaxations? How can we generate better ones?
- This connects to a rich theory of lift-and-project procedures, and of extensions of polytopes...

The End

Thank You