Positive Semidefinite Rank

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Section 1

Definition and Basic Properties

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The smallest size of a semidefinite factorization is defined to be the positive semidefinite rank of M, $\mathrm{rank}_{\mathrm{psd}}(M)$

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$$\operatorname{rank}_{\mathrm{psd}}(MN) \leq \min(\operatorname{rank}_{\mathrm{psd}}(M), \operatorname{rank}_{\mathrm{psd}}(N)).$$

Basic Bounds

Dimension Bounds

If $M \in \mathbb{R}_+^{p \times q}$ is a nonnegative matrix, then

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$$\operatorname{rank}_{\mathrm{psd}}\left(\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ * & A_2 & \ddots & 0 \\ * & * & \ddots & 0 \\ * & * & \cdots & A_n \end{bmatrix}\right) \ge \sum_{i=1}^{n} \operatorname{rank}_{\mathrm{psd}}(A_i).$$

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In particular $\operatorname{rank}_{\text{psd}}(I_n) = n$.

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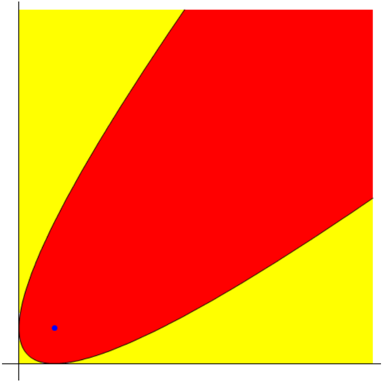

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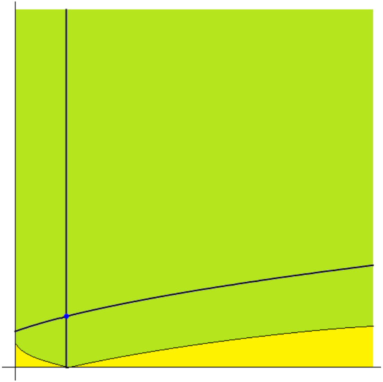

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$$\mathcal{P}_{p,q,k} := \{M \in \mathbb{R}_+^{p\times q} \mid \mathrm{rank}_{\mathrm{psd}}(M) \leq k\}$$

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Even in the case $(3, 3, 2)$ the precise description is not completely known.

Section 2

Geometric Motivation

Semidefinite Representations

A semidefinite representation of size k of a polytope P is a description

$$P = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists y \text{ s.t. } A_0 + \sum A_i x_i + \sum B_i y_i \succeq 0 \right\}$$

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This tells us how hard it is to optimize over P using semidefinite programming.

The Square

The $0/1$ square is the projection onto x_1 and x_2 of

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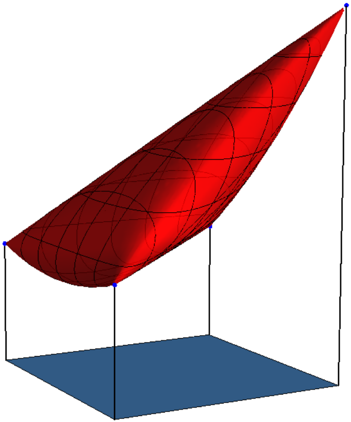

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Let P be a polytope with facets given by $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

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Theorem (G.-Parrilo-Thomas 2011)

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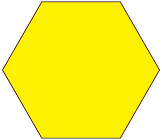
Lemma (Gillis-Glineur 12)

All nonnegative matrices of rank $n+1$ can be seen as generalized slack matrices of polytopes of dimension n.

The Hexagon

Consider the regular hexagon.



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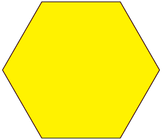

Its 6×6 slack matrix.

$$\begin{bmatrix} 0 & 0 & 2 & 4 & 4 & 2 \\ 2 & 0 & 0 & 2 & 4 & 4 \\ 4 & 2 & 0 & 0 & 2 & 4 \\ 4 & 4 & 2 & 0 & 0 & 2 \\ 2 & 4 & 4 & 2 & 0 & 0 \\ 0 & 2 & 4 & 4 & 2 & 0 \end{bmatrix}$$

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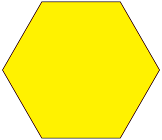

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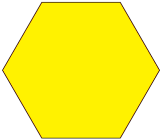

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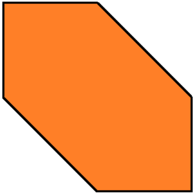

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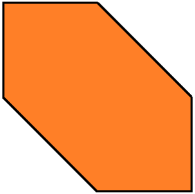

$$H = \left\{ (x_1, x_2) : \begin{bmatrix} 1 & x_1 & x_2 & x_1 + x_2 \\ x_1 & 1 & y_1 & y_2 \\ x_2 & y_1 & 1 & y_3 \\ x_1 + x_2 & y_2 & y_3 & 1 \end{bmatrix} \succeq 0 \right\}$$

Section 3

Computing Semidefinite Rank

Low rank cases

Rank 1

$\text{rank}\,(M) = 1 \Leftrightarrow \text{rank}_{\text{psd}}\,(M) = 1$

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Can we say more?

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Can we say more?

Let M_n be the (rank 3) slack matrix of a regular n-gon then $r_n = \text{rank}_{\text{psd}}\,(M_n) \longrightarrow +\infty$.

Semidefinite rank 2

If $\operatorname{rank}(M) > 3$ then $\operatorname{rank}_{\mathrm{psd}}(M) > 2$, so we need only to study rank 3 matrices.

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Let $M = S_{PQ}$ with $\operatorname{rank}(M) = 3$ then $\operatorname{rank}_{\mathrm{psd}}(M) = 2$ if and only if there is an ellipse E with $P \subseteq E \subseteq Q$.

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Convex Formulation

Let $P = \text{conv}(x_1, \cdots, x_n)$ and $Q = \{x : Gx \leq h\}$ then $\text{rank}_{\text{psd}}(S_{PQ}) = 2$ iff there exist A, b, c such that:

1. $A \succeq 0, \; \text{trace}(A) = 1$
2. $\begin{bmatrix} x_j \\ 1 \end{bmatrix}^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \begin{bmatrix} x_j \\ 1 \end{bmatrix} \leq 0 \quad \forall j$
3. $\exists \lambda_i \geq 0 \; : \; \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \succeq \lambda_i \begin{bmatrix} 0 & g_i^T/2 \\ g_i/2 & -h_i \end{bmatrix} \quad \forall i$

Example

$$M = \begin{bmatrix} 1+a & 1+b & 1-a & 1-b \\ 1-a & 1+b & 1+a & 1-b \\ 1-a & 1-b & 1+a & 1+b \\ 1+a & 1-b & 1-a & 1+b \end{bmatrix}$$

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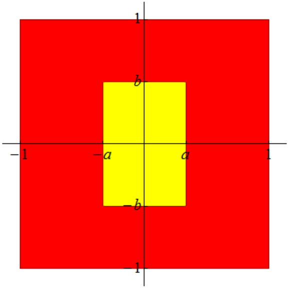

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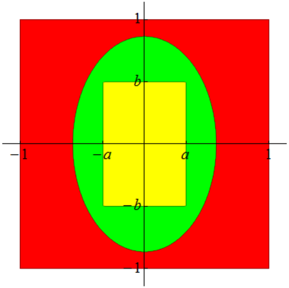

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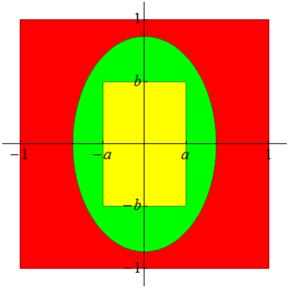

$$\operatorname{rank}_{\mathrm{psd}} M = \begin{cases} 3 & \text{if } a^2+b^2 > 1 \\ 2 & \text{if } 0 < a^2+b^2 \le 1 \\ 1 & \text{if } a = b = 0 \end{cases}$$

General case

A similar geometric picture holds more generally, and can be used to show general complexity results.

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Theorem

Let $M \in \mathbb{R}_+^{n\times m}$ with $\operatorname{rank}(M) = \binom{k+1}{2}$*. Checking if* $\operatorname{rank}_{\text{psd}} = k$ *can be solved in time $(nm)^{O(k^5)}$. In particular, for fixed k it is solvable in polynomial time.*

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- Is deciding $\operatorname{rank}_{\text{psd}}(M) < \min\{p, q\}$ for a $p \times q$ matrix NP-hard?

Section 4

Related Ranks

Nonnegative Rank

Let M be an m by n nonnegative matrix.

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The nonnegative rank can be seen as the semidefinite rank where we restrict our matrices to be diagonal. In particular

$$\text{rank}_{\text{psd}}(M) \leq \text{rank}_+(M).$$

Hadamard Square Root Rank

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Further thoughts on the square-root rank

0/1 matrices

If $M \in \{0,1\}^{n \times m}$ then $\mathrm{rank}_{\mathrm{psd}}(M) \leq \mathrm{rank}_{H}(M) \leq \mathrm{rank}(M)$.

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If M has at most k distinct entries, $\mathrm{rank}_{\mathrm{psd}}(M) \leq \binom{k-1+\mathrm{rank}(M)}{k-1}$

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$$\operatorname{rank}_H \begin{bmatrix} 1 & 0 & \cdots & 0 & a_1^2 \\ 0 & 1 & \ddots & 0 & a_2^2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_n^2 \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix} = n \text{ iff } \{a_1, ..., a_n\} \text{ can be partitioned}$$

Example: Euclidean Distance Matrices

Consider the $n \times n$ matrix *Euclidean distance matrix* M_n whose (i, j)-entry is $(i - j)^2$.

$$M_n = \begin{bmatrix} 0 & 1 & 4 & \cdots & (n-1)^2 \\ 1 & 0 & 1 & \ddots & (n-2)^2 \\ 4 & 1 & 0 & \ddots & (n-3)^2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ (n-1)^2 & (n-2)^2 & (n-3)^2 & \cdots & 0 \end{bmatrix}$$

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$\mathrm{rank}_{\mathrm{psd}}\,(M_n) = \mathrm{rank}_{\,H}(M_n) = 2$, while $\mathrm{rank}_{\,+}(M_n) \geq \log_2(n)$.

Example: The Prime Matrices

Let $n_1, n_2, n_3, \ldots$ be an increasing sequence such that $2n_k - 1$ is prime for each k. Define a $k \times k$ matrix Q^k such that $Q^k_{ij} = n_i + n_j - 1$.

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$\operatorname{rank}(Q^n) = 2 \Rightarrow \operatorname{rank}_{\text{psd}}(Q^n) = \operatorname{rank}_+(Q^n) = 2.$

However $\operatorname{rank}_H(Q^n) = n$.

Section 5

Other Interesting Topics That I Have Not Enough Time To Talk About In Length

Symmetric PSD factorizations

Definition

A symmetric matrix $M \in \mathrm{Sym}^n$ is *completely psd* if there exist $A_1, \ldots, A_n \in \mathrm{PSD}^k$ such that $M_{ij} = \langle A_i, A_j \rangle$.

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- Is CP^n closed?

Dependency on the field

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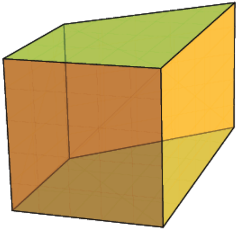
$$M = \begin{pmatrix} 0 & 0 & 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \end{pmatrix}$$



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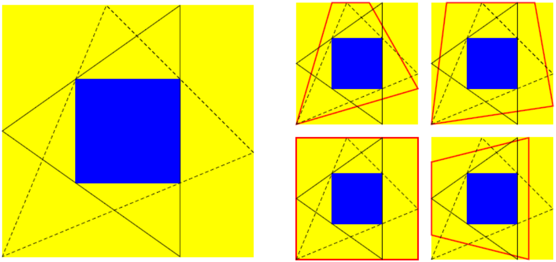

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Section 6

Conclusion

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The end

Survey coming soon!!!

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In the meantime:

J. Briët, D. Dadush, and S. Pokutta.
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Thank you