Semidefinite lifts of polytopes

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Semidefinite Representations

A semidefinite representation of size k of a polytope P is a description

$$P = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists y \text{ s.t. } A_0 + \sum A_i x_i + \sum B_i y_i \succeq 0 \right\}$$

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This tells us how hard it is to optimize over P using semidefinite programming.

The Square

The $0/1$ square is the projection onto x_1 and x_2 of

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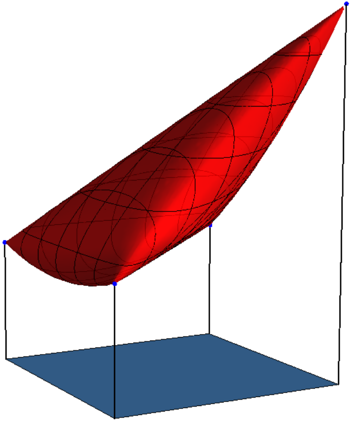

Slack Matrix

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\begin{array}{r} x \geq 0 \\ y \geq 0 \\ z \geq 0 \\ 1 - x \geq 0 \\ 1 - y \geq 0 \\ 1 - z \geq 0 \end{array} & \left[\begin{array}{cccccccc} \; & \; & \; & \; & \; & \; & \; & \; \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{array} \right]
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Semidefinite Factorizations

Let M be a m by n nonnegative matrix.

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Theorem (G.-Parrilo-Thomas 2011)

A polytope P has a semidefinite representation of size k if and only if its slack matrix has a PSD_k-factorization.

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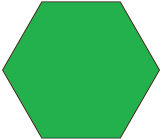
The psd rank of a polytope P is defined as

$$\mathrm{rank}_{psd}(P) := \mathrm{rank}_{psd}(S_P).$$

The Hexagon

Consider the regular hexagon.



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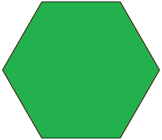

It has a 6×6 slack matrix.

$$\begin{bmatrix} 0 & 0 & 2 & 4 & 4 & 2 \\ 2 & 0 & 0 & 2 & 4 & 4 \\ 4 & 2 & 0 & 0 & 2 & 4 \\ 4 & 4 & 2 & 0 & 0 & 2 \\ 2 & 4 & 4 & 2 & 0 & 0 \\ 0 & 2 & 4 & 4 & 2 & 0 \end{bmatrix}$$

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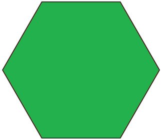

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$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix},$$

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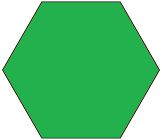

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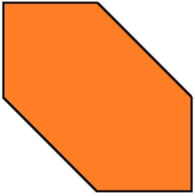

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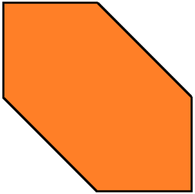

$$H = \left\{ (x_1, x_2) : \begin{bmatrix} 1 & x_1 & x_2 & x_1 + x_2 \\ x_1 & 1 & y_1 & y_2 \\ x_2 & y_1 & 1 & y_3 \\ x_1 + x_2 & y_2 & y_3 & 1 \end{bmatrix} \succeq 0 \right\}$$

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We want to make "small" $= d+1$.

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$\sqrt[H+]{M}$ is the nonnegative Hadamard square root of M.

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Corollary

For $0/1$ matrices

$$\text{rank}_{\text{psd}}(M) \leq \text{rank}_H(M) \leq \text{rank}(M).$$

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Polytopes with minimal representations

We can already recover an older result obtained originally using sums of squares.

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But we can say much more.

Main Theorem

Let P have dimension d. Then

$$\text{rank}_{\text{psd}}(P) = d+1 \Leftrightarrow \text{rank}_H(S_P) = d+1.$$

Properties of SDP-minimal Polytopes

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- Pyramids over SDP-minimal polytopes are SDP-minimal.

Results in $\mathbb{R}^2$

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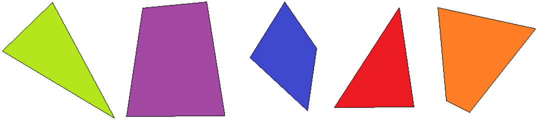
Proposition

A convex polygon is SDP-minimal if and only if it is a triangle or a quadrilateral.



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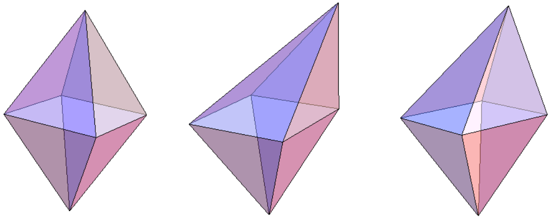

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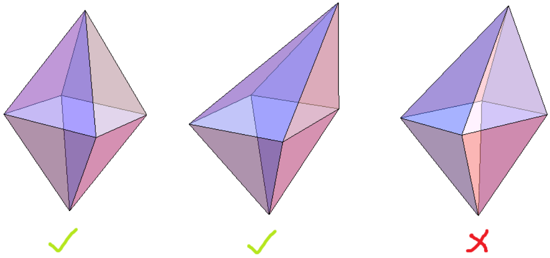

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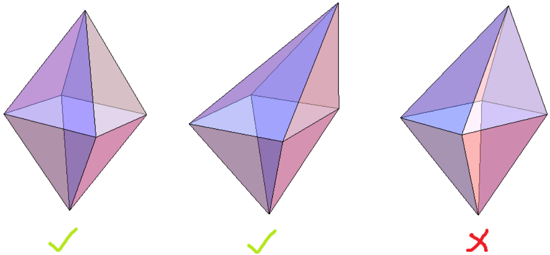

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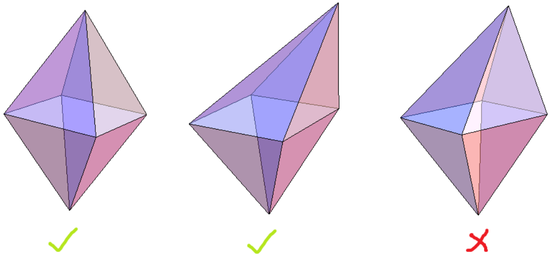

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It also suggests some underlying matroid characterization.

Open questions

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- Classification of SDP-minimal polyhedra.
- Better Algebraic/Geometrical characterization of SDP-minimality.

For more information

Polytopes of Minimum Positive Semidefinite Rank - Gouveia, Robinson and Thomas - arXiv:1205.5306

Lifts of convex sets and cone factorizations - Gouveia, Parrilo and Thomas - arXiv:1111.3164

Lifts and Factorizations of Convex Sets - Semiplenary talk by Rekha this afternoon.

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Thank you