A new SDP approach to the Max-Cut problem

J. Gouveia[1] M. Laurent[2] P. Parrilo[3] R. Thomas[1]

[1] University of Washington

[2]CWI, Amsterdam

[3]Massachusetts Institute of Technology

25th April '09 / AMS Spring Western Section Meeting

The Stable Set Problem

The Stable Set Problem

Given a graph $G = (\{1, ..., n\}, E)$ we'll denote by S_G the collection of the **characteristic vectors** of all the **stable sets** of G.

The Stable Set Problem

Given a graph $G = (\{1, ..., n\}, E)$ we'll denote by S_G the collection of the **characteristic vectors** of all the **stable sets** of G. We define the **stable set polytope** of G, $\mathrm{STAB}(G)$, as the convex hull of S_G.

The Stable Set Problem

Given a graph $G = (\{1, ..., n\}, E)$ we'll denote by S_G the collection of the **characteristic vectors** of all the **stable sets** of G. We define the **stable set polytope** of G, $\mathrm{STAB}(G)$, as the convex hull of S_G.

Stable Set Problem - LP Formulation

Given a graph $G = (\{1, ..., n\}, E)$ and a weight vector $\omega \in \mathbb{R}^n$, solve the linear program

$$\alpha(G, \omega) := \max_{x \in \mathrm{STAB}(G)} \langle \omega, x \rangle .$$

Definition of Theta Body

Definition (Lovász $\sim$ 1980)

Given a graph $G = (\{1, ..., n\}, E)$ we define its theta body, $\mathrm{TH}(G)$, as the set of all vectors $x \in \mathbb{R}^n$ such that

$$\begin{bmatrix} 1 & x^t \\ x & U \end{bmatrix} \succeq 0$$

for some symmetric $U \in \mathbb{R}^{n \times n}$ with $\mathrm{diag}(U) = x$ and $U_{ij} = 0$ for all $(i, j) \in E$.

Definition of Theta Body

Definition (Lovász ∼ 1980)

Given a graph $G = (\{1, ..., n\}, E)$ we define its theta body, $\mathrm{TH}(G)$, as the set of all vectors $x \in \mathbb{R}^n$ such that

$$\begin{bmatrix} 1 & x^t \\ x & U \end{bmatrix} \succeq 0$$

for some symmetric $U \in \mathbb{R}^{n\times n}$ with $\mathrm{diag}(U) = x$ and $U_{ij} = 0$ for all $(i, j) \in E$.

Theorem (Lovász ∼ 1980)

The relaxation is tight, i.e. $TH(G) = STAB(G)$, if and only if the graph G is perfect.

k-Sums of Squares

Let $I \subseteq \mathbb{R}[\mathbf{x}]$ be a polynomial ideal.

k-Sums of Squares

Let $I \subseteq \mathbb{R}[\mathbf{x}]$ be a polynomial ideal.
We say a polynomial f is k**-sos modulo the ideal** I if and only if

$$f \equiv (h_1^2 + h_2^2 + ... + h_m^2) \mod I,$$

for some polynomials $h_1, ..., h_m$ with degree less or equal k.

k-Sums of Squares

Let $I \subseteq \mathbb{R}[\mathbf{x}]$ be a polynomial ideal.
We say a polynomial f is k**-sos modulo the ideal** I if and only if

$$f \equiv (h_1^2 + h_2^2 + \ldots + h_m^2) \mod I,$$

for some polynomials $h_1, \ldots, h_m$ with degree less or equal k.

In particular, for any $\mathbf{p}$ in the zero set $\mathcal{Z}(I)$ we have

$$f(\mathbf{p}) = h_1^2(\mathbf{p}) + \ldots + h_m^2(\mathbf{p}) \geq 0,$$

so any k-sos polynomial is a nonnegative on the zero-set of the ideal.

Connection to Algebra

Theorem (Lovász $\sim$ 1993)

$TH(G)$ equals the intersection of all half-spaces

$$H_f = \{x \in \mathbb{R}^n : f(x) \geq 0\}$$

where f ranges over all affine polynomials that are 1-sos modulo $\mathcal{I}(S_G)$.

Connection to Algebra

Theorem (Lovász $\sim$ 1993)

$TH(G)$ equals the intersection of all half-spaces

$$H_f = \{x \in \mathbb{R}^n : f(x) \geq 0\}$$

where f ranges over all affine polynomials that are 1-sos modulo $\mathcal{I}(S_G)$.

This definition does not depend directly on the combinatorics of the graph, but only on the ideal $\mathcal{I}(S_G)$.

Theta Bodies of Ideals

Definition

Given an ideal $I \subset \mathbb{R}[x_1, ..., x_n]$ we define is k-th theta body, $\mathrm{TH}_k(I)$, as the intersection of all half-spaces

$$H_f = \{x \in \mathbb{R}^n : f(x) \geq 0\}$$

where f ranges over all affine polynomials that are k-sos modulo I.

Theta Bodies of Ideals

Definition

Given an ideal $I \subset \mathbb{R}[x_1, ..., x_n]$ we define is k-th theta body, $\mathrm{TH}_k(I)$, as the intersection of all half-spaces

$$H_f = \{x \in \mathbb{R}^n : f(x) \geq 0\}$$

where f ranges over all affine polynomials that are k-sos modulo I.

Remarks:

Theta Bodies of Ideals

Definition

Given an ideal $I \subset \mathbb{R}[x_1, ..., x_n]$ we define is k-th theta body, $\mathrm{TH}_k(I)$, as the intersection of all half-spaces

$$H_f = \{x \in \mathbb{R}^n : f(x) \geq 0\}$$

where f ranges over all affine polynomials that are k-sos modulo I.

Remarks:

- $\overline{\mathrm{conv}(\mathcal{Z}(I))} \subseteq \cdots \subseteq \mathrm{TH}_k(I) \subseteq \mathrm{TH}_{k-1}(I) \subseteq \cdots \subseteq \mathrm{TH}_1(I)$.

Theta Bodies of Ideals

Definition

Given an ideal $I \subset \mathbb{R}[x_1, ..., x_n]$ we define is k-th theta body, $\mathrm{TH}_k(I)$, as the intersection of all half-spaces

$$H_f = \{x \in \mathbb{R}^n : f(x) \geq 0\}$$

where f ranges over all affine polynomials that are k-sos modulo I.

Remarks:

- $\overline{\mathrm{conv}(\mathcal{Z}(I))} \subseteq \cdots \subseteq \mathrm{TH}_k(I) \subseteq \mathrm{TH}_{k-1}(I) \subseteq \cdots \subseteq \mathrm{TH}_1(I)$.
- If $S \subset \mathbb{R}^n$ is a finite set and $I = \mathcal{I}(S)$ then for some k, we have $\mathrm{TH}_k(I) = \mathrm{conv}(S)$.

Combinatorial Moment Matrices

Let I be a polynomial ideal and

$$\mathcal{B} = \{1 = f_0, x_1 = f_1, ..., x_n = f_n, f_{n+1}, ...\}$$

be a basis of $\mathbb{R}[\mathbf{x}]/I$ and $\mathcal{B}_k = \{f_i : \deg(f_i) \leq k\}$ for all k.

Combinatorial Moment Matrices

Let I be a polynomial ideal and

$$\mathcal{B} = \{1 = f_0, x_1 = f_1, ..., x_n = f_n, f_{n+1}, ...\}$$

be a basis of $\mathbb{R}[\mathbf{x}]/I$ and $\mathcal{B}_k = \{f_i : \deg(f_i) \leq k\}$ for all k. Consider the polynomial vector $f^k(\mathbf{x}) = (f_i(\mathbf{x}))_{f_i \in \mathcal{B}_k}$ then

$$(f^k(\mathbf{x}))(f^k(\mathbf{x}))^t = \sum_{f_i \in \mathcal{B}} A_i f_i(\mathbf{x})$$

for some symmetric matrices A_i.

Combinatorial Moment Matrices

Let I be a polynomial ideal and

$$\mathcal{B} = \{1 = f_0, x_1 = f_1, ..., x_n = f_n, f_{n+1}, ...\}$$

be a basis of $\mathbb{R}[\mathbf{x}]/I$ and $\mathcal{B}_k = \{f_i : \deg(f_i) \leq k\}$ for all k. Consider the polynomial vector $f^k(\mathbf{x}) = (f_i(\mathbf{x}))_{f_i \in \mathcal{B}_k}$ then

$$(f^k(\mathbf{x}))(f^k(\mathbf{x}))^t = \sum_{f_i \in \mathcal{B}} A_i f_i(\mathbf{x})$$

for some symmetric matrices A_i. Given a vector y indexed by the elements in $\mathcal{B}$ we define the **combinatorial moment matrix** of y as

$$M_{\mathcal{B},k}(y) = \sum_{f_i \in \mathcal{B}} A_i y_{f_i}.$$

Theta Bodies and Moment Matrices

Theorem (GPT)

Let I be a polynomial ideal and $\mathcal{B} = \{1, x_1, ..., x_n, ...\}$ a basis for $\mathbb{R}[\mathbf{x}]/I$. Let

$$\mathcal{M}_{\mathcal{B},k}(I) = \{y \in \mathbb{R}^{\mathcal{B}} : y_0 = 1; M_{\mathcal{B},k}(y) \succeq 0\}$$

then

$$TH_k(I) = \overline{\pi_{\mathbb{R}^n}(\mathcal{M}_{\mathcal{B},k}(I))}$$

where $\pi_{\mathbb{R}^n} : \mathbb{R}^{\mathcal{B}} \to \mathbb{R}^n$ is just the projection over the coordinates indexed by the degree one monomials.

Zero-dimensional Varieties

Definition

We call an ideal $\mathbf{TH}_k$**-exact** if $\mathrm{TH}_k(I) = \overline{\mathrm{conv}(\mathcal{Z}(I))}$.

Zero-dimensional Varieties

Definition

We call an ideal $\mathbf{TH}_k$**-exact** if $\mathrm{TH}_k(I) = \overline{\mathrm{conv}(\mathcal{Z}(I))}$.

A full characterization is possible for $k = 1$ in the case of vanishing ideals of finite sets in $\mathbb{R}^n$.

Zero-dimensional Varieties

Definition

We call an ideal $\mathbf{TH}_k$**-exact** if $\mathrm{TH}_k(I) = \overline{\mathrm{conv}(\mathcal{Z}(I))}$.

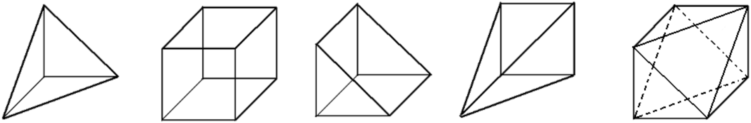
A full characterization is possible for $k = 1$ in the case of vanishing ideals of finite sets in $\mathbb{R}^n$.

Theorem (GPT)

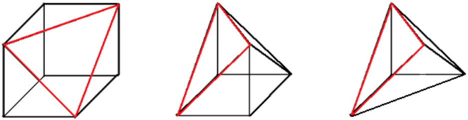
Let $S \subset \mathbb{R}^n$ be finite then $\mathcal{I}(S)$ is TH_1-exact if and only if for every facet defining hyperplane H of the polytope $conv(S)$ we have a parallel translate H' of H such that $S \subseteq H' \cup H$.

Examples in $\mathbb{R}^3$

TH_1-exact



Not TH_1-exact



The Max-Cut Problem

Definition

Given a graph $G = (V, E)$ and a partition V_1, V_2 of V the set C of edges between V_1 and V_2 is called a **cut**.

The Max-Cut Problem

Definition

Given a graph $G = (V, E)$ and a partition V_1, V_2 of V the set C of edges between V_1 and V_2 is called a **cut**.

The Problem

Given edge weights α we want to find which cut C maximizes

$$\alpha(C) := \sum_{(i,j) \in C} \alpha_{i,j}.$$

The Cut Polytope

Definition

The cut polytope of G, $\mathrm{CUT}(G)$, is the convex hull of the characteristic vectors $\chi_C \subseteq \mathbb{R}^E$ of the cuts of G, where $(\chi_C)_{ij} = -1$ if $(i, j) \in C$ and 1 otherwise.

The Cut Polytope

Definition

The cut polytope of G, $\mathrm{CUT}(G)$, is the convex hull of the characteristic vectors $\chi_C \subseteq \mathbb{R}^E$ of the cuts of G, where $(\chi_C)_{ij} = -1$ if $(i, j) \in C$ and 1 otherwise.

LP formulation

Given a vector $\alpha \in \mathbb{R}^E$ solve the optimization problem

$$\mathrm{mcut}(G, \alpha) = max_{x \in \mathrm{CUT}(G)} \frac{1}{2} \langle \alpha, \mathbf{1} - x \rangle .$$

Computing the ideal

Let I_G be the vanishing ideal of the characteristic vectors of the cuts of $G = (V, E)$.

Computing the ideal

Let I_G be the vanishing ideal of the characteristic vectors of the cuts of $G = (V, E)$.
Given an even set $T \subseteq V$ we define a T**-join** to be a subgraph of G with odd degree precisely in the vertices of T.

Computing the ideal

Let I_G be the vanishing ideal of the characteristic vectors of the cuts of $G = (V, E)$.
Given an even set $T \subseteq V$ we define a T**-join** to be a subgraph of G with odd degree precisely in the vertices of T. Let F_T be a T-join with a minimal number of edges d_T.

Computing the ideal

Let I_G be the vanishing ideal of the characteristic vectors of the cuts of $G = (V, E)$.
Given an even set $T \subseteq V$ we define a T**-join** to be a subgraph of G with odd degree precisely in the vertices of T. Let F_T be a T-join with a minimal number of edges d_T.

Theorem

If G is connected then the set

$$\{x_e^2 - 1 : \ e \in E)\} \cup \{1 - \mathbf{x}^A : \ A \subseteq E, A \textit{ circuit in } G\}$$

generates I_G, *and*

$$\mathcal{B} := \{\mathbf{x}^{F_T} \ : \ T \subseteq [n], \, |T| \, \text{even}\}$$

is a basis for $\mathbb{R}[\mathbf{x}]/I_G$.

General Cut Theta body

Let $\mathcal{B}_k$ be the set of all even $T \subseteq V$ such that $d_T \leq k$.

Theorem

The set $TH_k(I_G)$ is given by

$$\left\{ y \in \mathbb{R}^E \ : \qquad \right\}.$$

General Cut Theta body

Let $\mathcal{B}_k$ be the set of all even $T \subseteq V$ such that $d_T \leq k$.

Theorem

The set $TH_k(I_G)$ is given by

$$\left\{ y \in \mathbb{R}^E \; : \; \begin{array}{c} \exists\, M \succeq 0,\; M \in \mathbb{R}^{|\mathcal{B}_k| \times |\mathcal{B}_k|} \text{ such that} \\ \\ \\ \end{array} \right\}.$$

General Cut Theta body

Let $\mathcal{B}_k$ be the set of all even $T \subseteq V$ such that $d_T \leq k$.

Theorem

The set $TH_k(I_G)$ is given by

$$\left\{ y \in \mathbb{R}^E \;:\; \begin{array}{l} \exists\, M \succeq 0,\ M \in \mathbb{R}^{|\mathcal{B}_k| \times |\mathcal{B}_k|} \text{ such that} \\ M_{T,T} = 1 \ \forall\ T \in \mathcal{B}_k, \\ \\ \end{array} \right\}.$$

General Cut Theta body

Let $\mathcal{B}_k$ be the set of all even $T \subseteq V$ such that $d_T \leq k$.

Theorem

The set $TH_k(I_G)$ is given by

$$\left\{ y \in \mathbb{R}^E \;:\; \begin{array}{l} \exists\, M \succeq 0,\ M \in \mathbb{R}^{|\mathcal{B}_k| \times |\mathcal{B}_k|} \text{ such that} \\ M_{T,T} = 1 \ \forall\ T \in \mathcal{B}_k, \\ M_{e,\emptyset} = y_e \ \forall e \in E \end{array} \right\}.$$

General Cut Theta body

Let $\mathcal{B}_k$ be the set of all even $T \subseteq V$ such that $d_T \leq k$.

Theorem

The set $TH_k(I_G)$ is given by

$$\left\{ y \in \mathbb{R}^E \; : \; \begin{array}{l} \exists\, M \succeq 0,\; M \in \mathbb{R}^{|\mathcal{B}_k| \times |\mathcal{B}_k|} \text{ such that} \\ M_{T,T} = 1 \;\forall\; T \in \mathcal{B}_k, \\ M_{e,\emptyset} = y_e \;\forall e \in E \\ M_{T,T'} = M_{R,R'} \text{ if } T \Delta T' = R \Delta R' \end{array} \right\}.$$

The First Cut Theta Body

Cut Theta Body

Given a graph $G = (V, E)$ the body $TH_1(I_G)$ is the set of all $x \in \mathbb{R}^E$ such that

$$\begin{bmatrix} 1 & x^t \\ x & U \end{bmatrix} \succeq 0$$

for some a symmetric $U \in \mathbb{R}^{E \times E}$ with

The First Cut Theta Body

Cut Theta Body

Given a graph $G = (V, E)$ the body $TH_1(I_G)$ is the set of all $x \in \mathbb{R}^E$ such that

$$\begin{bmatrix} 1 & x^t \\ x & U \end{bmatrix} \succeq 0$$

for some a symmetric $U \in \mathbb{R}^{E \times E}$ with $\mathrm{diag}(U) = \mathbf{1}$,

The First Cut Theta Body

Cut Theta Body

Given a graph $G = (V, E)$ the body $TH_1(I_G)$ is the set of all $x \in \mathbb{R}^E$ such that

$$\begin{bmatrix} 1 & x^t \\ x & U \end{bmatrix} \succeq 0$$

for some a symmetric $U \in \mathbb{R}^{E \times E}$ with $\mathrm{diag}(U) = \mathbf{1}$, if (e, f, g) is a triangle in G, $U_{e,f} = x_g$,

The First Cut Theta Body

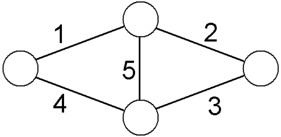
Cut Theta Body

Given a graph $G = (V, E)$ the body $TH_1(I_G)$ is the set of all $x \in \mathbb{R}^E$ such that

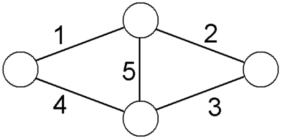
$$\begin{bmatrix} 1 & x^t \\ x & U \end{bmatrix} \succeq 0$$

for some a symmetric $U \in \mathbb{R}^{E \times E}$ with $\mathrm{diag}(U) = \mathbf{1}$, if (e, f, g) is a triangle in G, $U_{e,f} = x_g$, and if $\{e, f, g, h\}$ forms a 4-cycle $U_{e,f} = U_{g,h}$.

Example

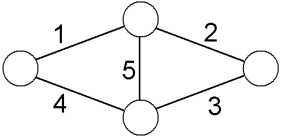


Example



$\mathrm{TH}_1(I_G)$ is the set of $x \in \mathbb{R}^5$ such that

Example

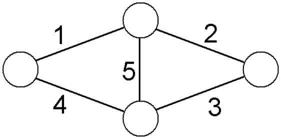


$\mathrm{TH}_1(I_G)$ is the set of $x \in \mathbb{R}^5$ such that

	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

$\succeq 0$

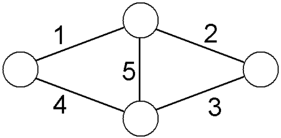
Example



$\mathrm{TH}_1(I_G)$ is the set of $x \in \mathbb{R}^5$ such that

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & & & & & \\ x_2 & & & & & \\ x_3 & & & & & \\ x_4 & & & & & \\ x_5 & & & & & \end{bmatrix} \end{array} \succeq 0$$

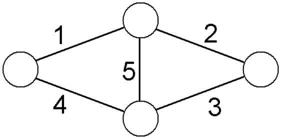
Example



$\mathrm{TH}_1(I_G)$ is the set of $x \in \mathbb{R}^5$ such that

$$
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{cccccc}
1 & x_1 & x_2 & x_3 & x_4 & x_5 \\
x_1 & 1 & & & & \\
x_2 & & 1 & & & \\
x_3 & & & 1 & & \\
x_4 & & & & 1 & \\
x_5 & & & & & 1
\end{array}\right]
\end{array}
\succeq 0
$$

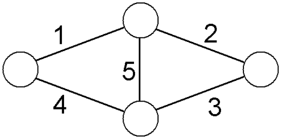
Example



$\mathrm{TH}_1(I_G)$ is the set of $x \in \mathbb{R}^5$ such that

$$
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{cccccc}
1 & x_1 & x_2 & x_3 & x_4 & x_5 \\
x_1 & 1 & & & {\color{red}?} & \\
x_2 & & 1 & & & \\
x_3 & & & 1 & & \\
x_4 & & & & 1 & \\
x_5 & & & & & 1
\end{array}\right]
\end{array}
\succeq 0
$$

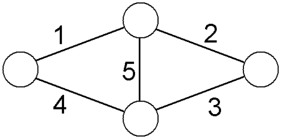
Example



$\mathrm{TH}_1(I_G)$ is the set of $x \in \mathbb{R}^5$ such that

$$
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{cccccc}
1 & x_1 & x_2 & x_3 & x_4 & x_5 \\
x_1 & 1 & & & x_5 & \\
x_2 & & 1 & & & \\
x_3 & & & 1 & & \\
x_4 & & & & 1 & \\
x_5 & & & & & 1
\end{array}\right]
\end{array}
\succeq 0
$$

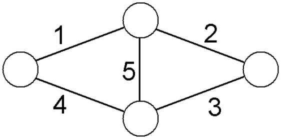
Example



$\mathrm{TH}_1(I_G)$ is the set of $x \in \mathbb{R}^5$ such that

$$
\begin{array}{c|cccccc|}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & 1 & x_1 & x_2 & x_3 & x_4 & x_5 \\
1 & x_1 & 1 & & & x_5 & x_4 \\
2 & x_2 & & 1 & & & \\
3 & x_3 & & & 1 & & \\
4 & x_4 & & & & 1 & \\
5 & x_5 & & & & & 1
\end{array} \succeq 0
$$

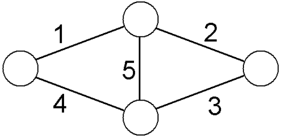
Example



$\mathrm{TH}_1(I_G)$ is the set of $x \in \mathbb{R}^5$ such that

$$
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{cccccc}
1 & x_1 & x_2 & x_3 & x_4 & x_5 \\
x_1 & 1 & & & x_5 & x_4 \\
x_2 & & 1 & x_5 & & \\
x_3 & & & 1 & & \\
x_4 & & & & 1 & \\
x_5 & & & & & 1
\end{array}\right]
\end{array}
\succeq 0
$$

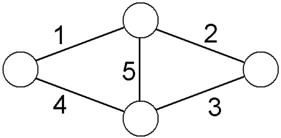
Example



$\mathrm{TH}_1(I_G)$ is the set of $x \in \mathbb{R}^5$ such that

$$
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix}
1 & x_1 & x_2 & x_3 & x_4 & x_5 \\
x_1 & 1 & & & x_5 & x_4 \\
x_2 & & 1 & x_5 & & x_3 \\
x_3 & & & 1 & & x_2 \\
x_4 & & & & 1 & x_1 \\
x_5 & & & & & 1
\end{bmatrix}
\end{array}
\succeq 0
$$

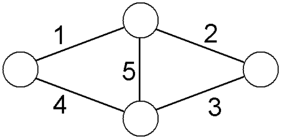
Example



$\mathrm{TH}_1(I_G)$ is the set of $x \in \mathbb{R}^5$ such that

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix}
1 & x_1 & x_2 & x_3 & x_4 & x_5 \\
x_1 & 1 & ? & & x_5 & x_4 \\
x_2 & & 1 & x_5 & & x_3 \\
x_3 & & & 1 & ? & x_2 \\
x_4 & & & & 1 & x_1 \\
x_5 & & & & & 1
\end{bmatrix}
\end{array} \succeq 0$$

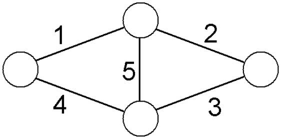
Example



$\mathrm{TH}_1(I_G)$ is the set of $x \in \mathbb{R}^5$ such that

$$
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{cccccc}
1 & x_1 & x_2 & x_3 & x_4 & x_5 \\
x_1 & 1 & y_1 & & x_5 & x_4 \\
x_2 & & 1 & x_5 & & x_3 \\
x_3 & & & 1 & y_1 & x_2 \\
x_4 & & & & 1 & x_1 \\
x_5 & & & & & 1
\end{array}\right]
\end{array}
\succeq 0
$$

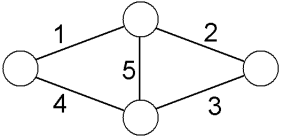
Example



$\mathrm{TH}_1(I_G)$ is the set of $x \in \mathbb{R}^5$ such that

$$
\begin{array}{c c}
 & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{bmatrix}
1 & x_1 & x_2 & x_3 & x_4 & x_5 \\
x_1 & 1 & y_1 & \color{red}{?} & x_5 & x_4 \\
x_2 & & 1 & x_5 & \color{red}{?} & x_3 \\
x_3 & & & 1 & y_1 & x_2 \\
x_4 & & & & 1 & x_1 \\
x_5 & & & & & 1
\end{bmatrix} \succeq 0
\end{array}
$$

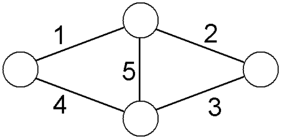
Example



$\mathrm{TH}_1(I_G)$ is the set of $x \in \mathbb{R}^5$ such that

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix} 1 & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & 1 & y_1 & y_2 & x_5 & x_4 \\ x_2 & & 1 & x_5 & y_2 & x_3 \\ x_3 & & & 1 & y_1 & x_2 \\ x_4 & & & & 1 & x_1 \\ x_5 & & & & & 1 \end{bmatrix} \end{array} \succeq 0$$

Example



$\mathrm{TH}_1(I_G)$ is the set of $x \in \mathbb{R}^5$ such that

$$\begin{array}{c c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1 & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & 1 & y_1 & y_2 & x_5 & x_4 \\ x_2 & y_1 & 1 & x_5 & y_2 & x_3 \\ x_3 & y_2 & x_5 & 1 & y_1 & x_2 \\ x_4 & x_5 & y_2 & y_1 & 1 & x_1 \\ x_5 & x_4 & x_3 & x_2 & x_1 & 1 \end{bmatrix} \succeq 0 \end{array}$$

Cut-Perfect Graphs

In analogy with the stable set results, it makes sense to have the following definition:

Cut-Perfect Graphs

In analogy with the stable set results, it makes sense to have the following definition:

Definition

We call a graph G **cut-perfect** if $\mathrm{TH}_1(I_G) = \mathrm{CUT}(G)$.

Cut-Perfect Graphs

In analogy with the stable set results, it makes sense to have the following definition:

Definition

We call a graph G **cut-perfect** if $\mathrm{TH}_1(I_G) = \mathrm{CUT}(G)$.

Using our characterization for TH_1-exact zero-dimensional ideals we get the following characterization, that answers a Lovász question.

Cut-Perfect Graphs

In analogy with the stable set results, it makes sense to have the following definition:

Definition

We call a graph G **cut-perfect** if $\mathrm{TH}_1(I_G) = \mathrm{CUT}(G)$.

Using our characterization for TH_1-exact zero-dimensional ideals we get the following characterization, that answers a Lovász question.

Theorem (GLPT)

A graph is cut-perfect if and only if it has no K_5 minor and no chordless cycle of size larger than 4.

Some remarks

- This relaxation is related to a previous relaxation by Monique Laurent which was derived using a different construction.

Some remarks

- This relaxation is related to a previous relaxation by Monique Laurent which was derived using a different construction.
- A cycle C_n is only $\mathrm{TH}_{\lceil n/4 \rceil}$-exact.

Some remarks

- This relaxation is related to a previous relaxation by Monique Laurent which was derived using a different construction.
- A cycle C_n is only $\mathrm{TH}_{\lceil n/4 \rceil}$-exact.
- The cycle problem can be avoided, if we add enough edges to the graph to start with.

Some remarks

- This relaxation is related to a previous relaxation by Monique Laurent which was derived using a different construction.
- A cycle C_n is only $\mathrm{TH}_{\lceil n/4 \rceil}$-exact.
- The cycle problem can be avoided, if we add enough edges to the graph to start with.
- This technique can in theory be applied to any combinatorial problem to derive hierarchies.

Some remarks

- This relaxation is related to a previous relaxation by Monique Laurent which was derived using a different construction.
- A cycle C_n is only $\mathrm{TH}_{\lceil n/4 \rceil}$-exact.
- The cycle problem can be avoided, if we add enough edges to the graph to start with.
- This technique can in theory be applied to any combinatorial problem to derive hierarchies. Results may vary.

The End

Thank You