Positive Semidefinite Rank

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with Hamza Fawzi (MIT), Pablo Parrilo (MIT), Richard Z. Robinson (U.Washington) and Rekha Thomas (U.Washington)

Section 1

Definition and Basic Properties

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The smallest size of a semidefinite factorization is denoted by **positive semidefinite rank** of M, $\text{rank}_{\text{psd}}(M)$

Basic Properties

Properties

The psd rank is:

(i) invariant under transpositions or nonnegative scalings;

(ii) subadditive;

(iii) at least $\approx \sqrt{2\text{rank}}$;

(iv) at most the smallest dimension of the matrix;

(v) $\cdots$

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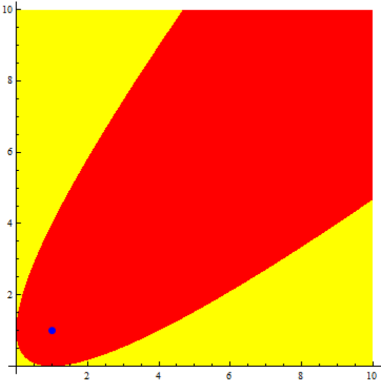

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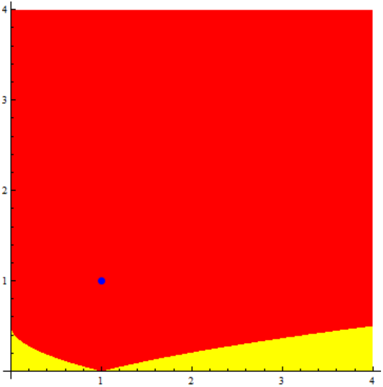

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$$\mathcal{P}_{p,q,k} := \{M \in \mathbb{R}_+^{p\times q} \mid \text{rank}_{\text{psd}}(M) \leq k\}$$

is a closed semialgebraic set inside the rank $\leq \binom{k+1}{2}$ variety.

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Even in the case $(3, 3, 2)$ the precise description is not easy.

Geometric Motivation

Given a polytope P described as a convex hull of n points and a polyhedron Q described by m inequalities with $P \subseteq Q$ we define $S_{P,Q} \subseteq \mathbb{R}_+^{n \times m}$ as the evaluation of the inequalities of Q at the points of P.

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Theorem (Semidefinite Yannakakis Theorem)

$\text{rank}_{\text{psd}}(S_{P,Q}) \leq k$ *if and only if there is a convex set* C *with an sdp representation of size* k *such that* $P \subseteq C \subseteq Q$.

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Lemma (Gillis-Glineur 12)

All nonnegative matrices of rank $n+1$ *can be seen as generalized slack matrices of polyhedra of dimension* n.

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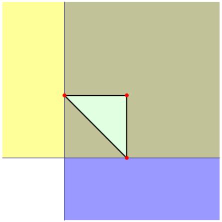

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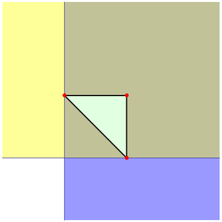

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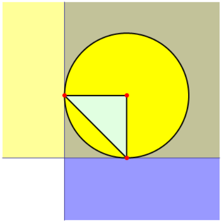

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Section 2

Computing Semidefinite Rank

Low (usual) rank cases

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Rank 3

$\text{rank}\,(M) = 3 \Rightarrow \text{rank}_{\text{psd}}\,(M) \geq 2$

Can we say more for rank 3?

Let M_n be the (rank 3) slack matrix of a regular n-gon then $\text{rank}_{\text{psd}}\,(M_n) \longrightarrow +\infty$.

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Let $M = S_{PQ}$ with $\operatorname{rank}(M) = 3$ then $\operatorname{rank}_{\text{psd}}(M) = 2$ if and only if there is an ellipse E with $P \subseteq E \subseteq Q$.

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Convex Formulation

Let $P = \text{conv}(x_1, \cdots, x_n)$ and $Q = \{x : Gx \leq h\}$ then $\text{rank}_{\text{psd}}\,(S_{PQ}) = 2$ iff there exist A, b, c such that:

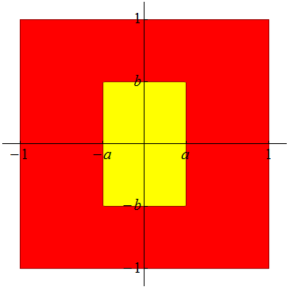
1. $A \succeq 0, \;\text{trace}(A) = 1$
2. $\begin{bmatrix} x_j \\ 1 \end{bmatrix}^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \begin{bmatrix} x_j \\ 1 \end{bmatrix} \leq 0 \quad \forall j$
3. $\exists \lambda_i \geq 0 \;:\; \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \succeq \lambda_i \begin{bmatrix} 0 & {g_i}^T/2 \\ g_i/2 & -h_i \end{bmatrix} \quad \forall i$

Example

$$M = \begin{bmatrix} 1+a & 1+b & 1-a & 1-b \\ 1-a & 1+b & 1+a & 1-b \\ 1-a & 1-b & 1+a & 1+b \\ 1+a & 1-b & 1-a & 1+b \end{bmatrix}$$

Example

$$\color{red}{M} = \begin{bmatrix} 1+\color{blue}{a} & 1+\color{green}{b} & 1-\color{blue}{a} & 1-\color{green}{b} \\ 1-\color{blue}{a} & 1+\color{green}{b} & 1+\color{blue}{a} & 1-\color{green}{b} \\ 1-\color{blue}{a} & 1-\color{green}{b} & 1+\color{blue}{a} & 1+\color{green}{b} \\ 1+\color{blue}{a} & 1-\color{green}{b} & 1-\color{blue}{a} & 1+\color{green}{b} \end{bmatrix}$$



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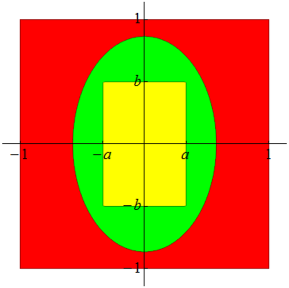

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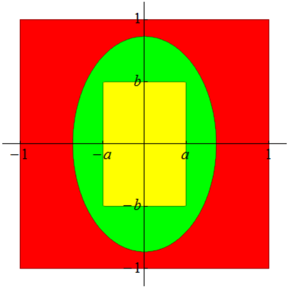

$$\mathrm{rank}_{\mathrm{psd}}\, M = \begin{cases} 3 & \text{if } a^2 + b^2 > 1 \\ 2 & \text{if } 0 < a^2 + b^2 \le 1 \\ 1 & \text{if } a = b = 0 \end{cases}$$

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Theorem

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- Is deciding $\operatorname{rank}_{\mathrm{psd}}(M) < \min\{p, q\}$ for a $p \times q$ matrix NP-hard?

Section 3

Square Root Rank

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Proposition

$\mathrm{rank}_{\sqrt{}}(M)$ rank corresponds to the semidefinite rank restricted to rank one factor matrices. In particular

$$\mathrm{rank}_{\mathrm{psd}}\,(M) \leq \mathrm{rank}_{\sqrt{}}(M).$$

How does the square root rank function look like?

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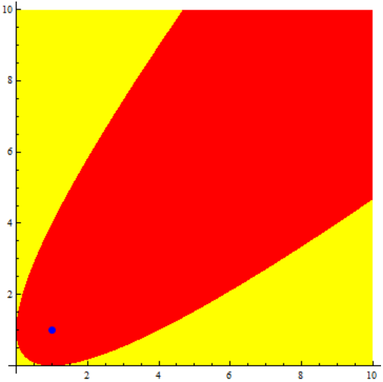

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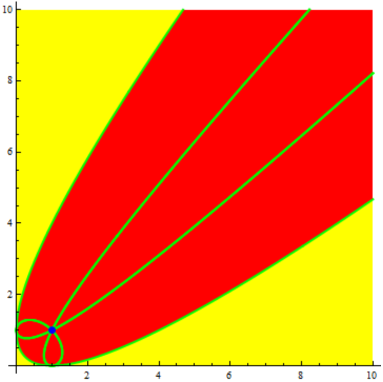

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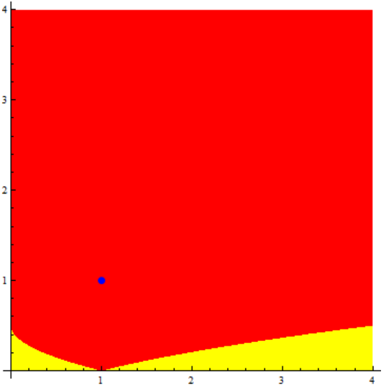

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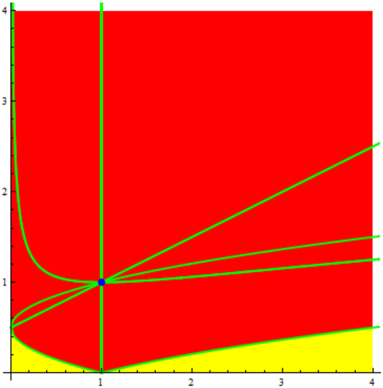

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Example: The Prime Matrices

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However $\operatorname{rank}_{\sqrt{\ }}(Q^k) = k$.

Further bad news on the square-root rank

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$$\mathrm{rank}_{\sqrt{\ }} \begin{bmatrix} 1 & 0 & \cdots & 0 & {a_1}^2 \\ 0 & 1 & \ddots & 0 & {a_2}^2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & {a_n}^2 \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix} = n \text{ iff } \{a_1, ..., a_n\} \text{ can be partitioned}$$

What works for square root rank

0/1 matrices

If $M \in \{0,1\}^{n\times m}$ then $\mathrm{rank}_{\mathrm{psd}}(M) \leq \mathrm{rank}_{\sqrt{\ }}(M) \leq \mathrm{rank}(M)$.

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Psd minimal polytopes

P a d-dimensional polytope, then $\text{rank}_{\text{psd}}(S_P) \geq d+1$ with equality if and only if $\text{rank}_{\sqrt{}}(S_P) = d+1$.

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$\mathbb{R}^2$ characterization

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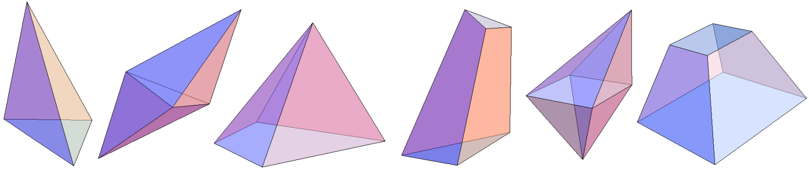

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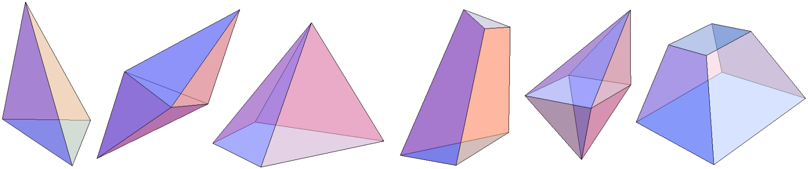

Kostya's talk for more news on that.

Section 4

Dependency on the field

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Theorem [Lee-Wei-de Wolf 14]

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Can the gap be 2?

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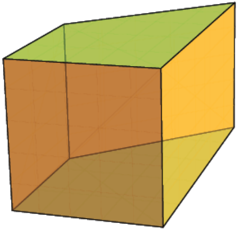
$$M = \begin{pmatrix} 0 & 0 & 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \end{pmatrix}$$



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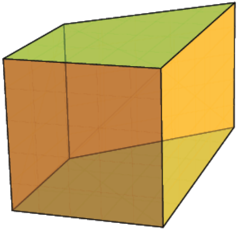

psd minimal $\Rightarrow$ rank one factors $\Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ cannot appear.

Section 5

Space of factorizations

Space of factorizations

Given M with $\text{rank}_{\text{psd}}(M) = k$ consider $\mathcal{SF}(M)$ the set of its *tgk* $\times$ k psd factorizations:

$$\mathcal{SF}(M) = \{(A_1, \cdots, A_n, B_1 \cdots, B_m) \in \text{PSD}_k^{m+n} \; : \; M_{ij} = \langle A_i, B_j \rangle, \forall i, j\}.$$

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For any $L \in \text{GL}(k)$ it is easy to see

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To $\mathcal{F}_k(M) := \mathcal{SF}(M)/\text{GL}(k)$ we call the **space of factorizations** of M.

Example

Recall that $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ has $\text{rank}_{\text{psd}}(M) = 2$.

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with $a \in [1/2, 1]$.

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Given $P \subseteq Q$, let $\mathcal{C}_k(P, Q)$ be the set of PSD$_k$-representable sets C such that $P \subseteq C \subseteq Q$.

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In fact, it is invariant with respect to the $\mathrm{GL}(k)$ action so we have a map

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from the space of factorizations to that of “sandwiched” sets.

Example revisited

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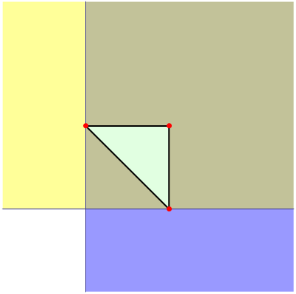

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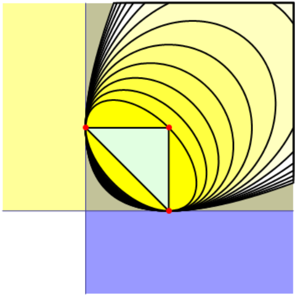

Here we actually have a one to one correspondence

Extremal case

Theorem
If $M = S_{P,Q}$ has $\operatorname{rank}_{\text{psd}}(M) = k$ and $\operatorname{rank}(M) = \binom{k+1}{2}$ then $\mathcal{F}_k(M)$ and $\mathcal{C}_k(P, Q)$ are homeomorphic.

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- $\mathcal{C}_4(H, H)$ has a single element;
- $\mathcal{F}_4(M)$ has at least 2 points.

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Proposition

For $\operatorname{rank}(M) = 3$, $\operatorname{rank}_{\text{psd}}(M) = 2$, $\mathcal{F}_2(M)$ is connected.

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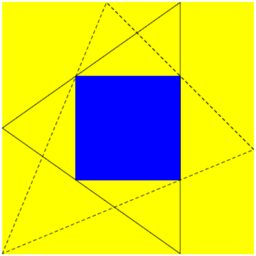

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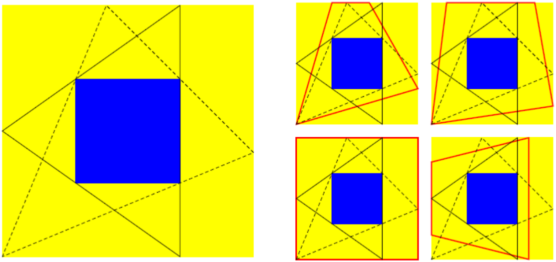

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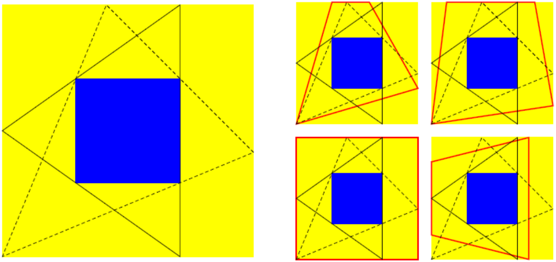

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Question

When is $\mathcal{F}_k(M)$ connected?

The end

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Thank you