Sums of squares in polynomial binary optimization

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Sums of squares certificates

Polynomial Optimization over Algebraic Varieties

$p_{\min} = \min p(x)$ over all x such that

$$x \in \{x \mid p_i(x) = 0,\ i = 1, ..., t\}.$$

Sums of squares certificates

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$p_{\min} = \min p(x)$ over all x such that

$$x \in \mathcal{V}(\langle p_1, \cdots, p_t \rangle) = \mathcal{V}(I).$$

Sums of squares certificates

Polynomial Optimization over Algebraic Varieties

$p_{\min} = \max \lambda$ such that

$$p(x) - \lambda \geq 0 \text{ for all } x \in \mathcal{V}(I).$$

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$p_{\min} \leq p_{\text{sos}} =$ max λ such that

$$p(x) - \lambda = \sum h_i^2 \bmod I.$$

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$p_{\min} \leq p_{\text{sos}} \leq p^k_{\text{sos}} = \max \lambda$ such that

$$p(x) - \lambda = \sum h_i^2 \bmod I, \quad \deg(h_i) \leq k.$$

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$p_{\min} \leq p_{\text{sos}} \leq p_{\text{sos}}^k = \max \lambda$ such that

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$$(p(x) - \lambda)g(x) \in \Sigma_k[I], \quad \text{for some positive } g(x)).$$

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$p_{\min} \leq p^{l,k}_{\text{sos}} = \max\ \lambda$ such that

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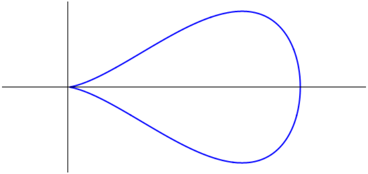
This is not a linear SDP anymore, but is still doable.

Example

Consider the teardrop curve given by $I = \langle x^4 - x^3 + y^2 \rangle$.



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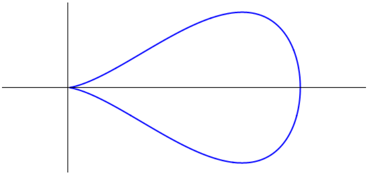

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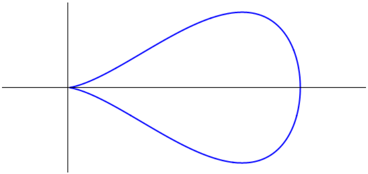

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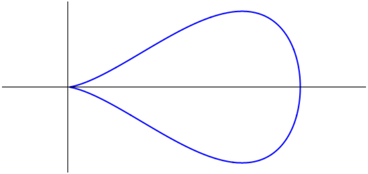

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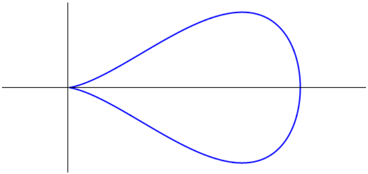

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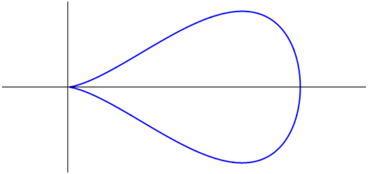

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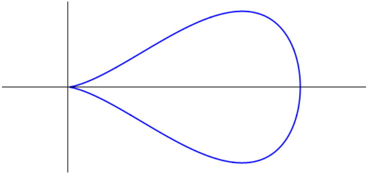

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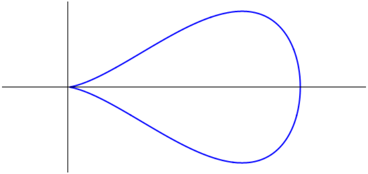

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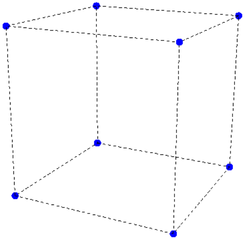
Multipliers make the relaxations less sensitive to singularities.

The n-cube

We are interested in the n-cube:

$$C_n = \{0,1\}^n = \{x \in \mathbb{R}^n \ : \ x_i^2 - x_i = 0, \ i = 1, \cdots, n\} = \mathcal{V}(I_n).$$



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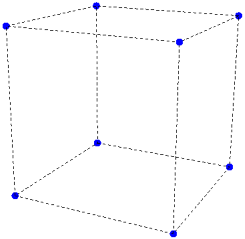

Cube C_3

S_n acts on C_n by permuting coordinates, and if p is symmetric, it will be completely characterized by its evaluation at the levels T_k of the cube:

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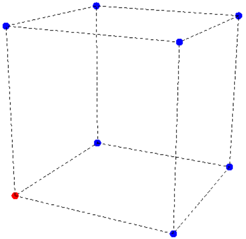

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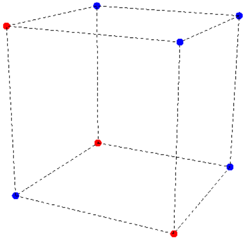

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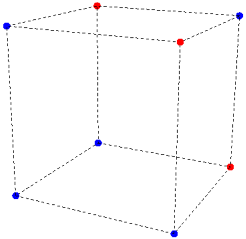

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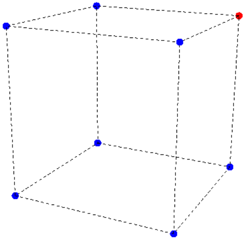

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Main Result 1 - Bad news

Let p be a symmetric square-free polynomial attaining its minimum over C_n at level T_k, with $\deg p \leq k \leq n/2$.

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The proof reduces to this lemma.

Lemma

If p has degree d and vanishes at T_k with $d \leq k \leq n - d$ then

$$p = (k - \sum x_i)q \mod I_n,$$

with $\deg q < \deg p$.

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& & & \oplus & & \oplus & & & & \oplus \\
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and is enough to check that M_j does not vanish at T_k.

Application 1 - MaxCut

Recall that the maxcut problem over K_n can be reduced to

Binary polynomial formulation of MaxCut

$$\max p(x) = \sum_{i \neq j} (1 - x_i) x_j \text{ s.t. } x \in C_n$$

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Note that p attains its maximum in C_n at T_k and T_{k+1}, which are not local maxima of p over $\mathbb{R}^n$.

First corollary of main result 1

For $n = 2k + 1$, $p_{\text{sos}}^{k-1,k} > p_{\max}$.

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For any k there is a degree 4 polynomial in $\mathbb{R}^{2k+1}$ for which $p_{\min} < p_{\text{sos}}^{k-2,k}$.

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This is proven by a perturbed extension of the polynomial on the previous example.

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Let p be a non constant quadratic polynomial in $\mathbb{R}^{2k+1}$, then $p_{\min} = p_{\text{sos}}^{k+1,k+2}$.

The proof is based in dimension counting.

Application - MaxCut revisited

Consider the weighted maxcut formulation.

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$$\max p_\omega(x) = \sum_{i \neq j} \omega_{ij}(1 - x_i)x_j \text{ s.t. } x \in C_n,$$

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A weaker version can now be proved.

Corollary of main result 2

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The End

Thank You