Sums of Squares on the Hypercube

Greg Blekherman [1] João Gouveia [2] James Pfeiffer [3]

[1]Georgia Tech

[2]Universidade de Coimbra

[3]University of Washington

Section 1

Introduction

Nonnegativity of a polynomial

Let $I \subseteq \mathbb{R}[x]$ be an ideal:

$$\mathcal{P}(I) = \{p \in \mathbb{R}[I] \ : \ p \text{ is nonnegative on } \mathcal{V}_{\mathbb{R}}(I)\}.$$

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Efficiently checking membership in $\mathcal{P}(I)$ is important for polynomial optimization.

A typical strategy is to approximate $\mathcal{P}(I)$ by

$$\Sigma(I) = \left\{ p \in \mathbb{R}[I] \;:\; p \equiv \sum_{i=1}^{t} h_i^2 \text{ for some } h_i \in \mathbb{R}[I] \right\},$$

and its truncations

$$\Sigma_k(I) = \left\{ p \in \mathbb{R}[I] \;:\; p \equiv \sum_{i=1}^{t} h_i^2 \text{ for some } h_i \in \mathbb{R}_k[I] \right\}.$$

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When are sums of squares enough?

Theorem (Hilbert 1888)

$\Sigma_k(\mathbb{R}^n) = \mathcal{P}_{2k}(\mathbb{R}^n)$ *if and only if* $n = 1$, $k = 1$ *or* $(n, k) = (2, 2)$.

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In other words, we want to bound the degrees of the denominators in the rational functions used.

Advantages and Disadavantages

Schmudgen's Positivstellensatz

If $\mathcal{V}_{\mathbb{R}}(I)$ is compact, p positive on $\mathcal{V}_{\mathbb{R}}(I)$ implies p is k-sos for some k.

No uniform bounds on how big can k be.

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- Checking k-rsosness is still an SDP feasibility problem.
- Optimizing over the set of all k-rsos polynomials is not as easy.

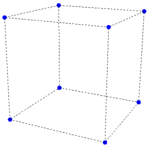
Section 2

Upper bounds on multipliers

The n-cube

We are interested in the n-cube:

$$C_n = \{0,1\}^n = \{x \in \mathbb{R}^n \ : \ x_i^2 - x_i = 0, \ i = 1, \cdots, n\} = \mathcal{V}(I_n).$$



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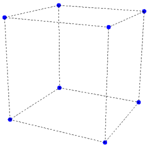

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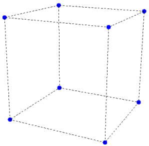

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Main Lemma

Let $\ell : \mathbb{R}[X]_{2d} \to \mathbb{R}$ be given by $\ell(f) = \sum_{v \in X} \mu_v f(v)$ with all $\mu_v \neq 0$. Suppose that ℓ is nonnegative on $\Sigma_d(X)$. Then

$$\#\{v \in X : \mu_v > 0\} \geq \dim \mathbb{R}[X]_d.$$

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Open Question: Is the increased degree needed?

Section 3

Lower bounds on hypercube multipliers

Hypercube

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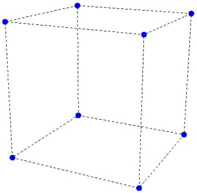

Cube C_3

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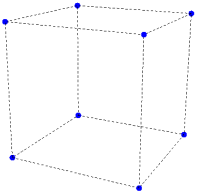

Cube C_3

S_n acts on C_n by permuting coordinates, and if p is symmetric, it will be completely characterized by its evaluation at the levels T_k of the cube:

$$T_k = \{x \in C_n \ : \ \sum x_i = k\}.$$

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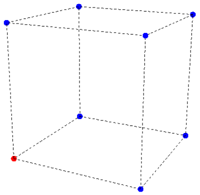

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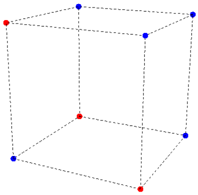

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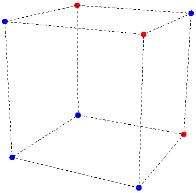

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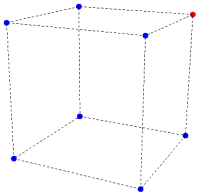

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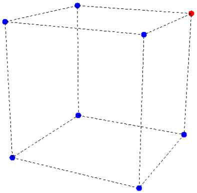

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Symmetric polynomials appear naturally in combinatorial optimization, and we want lower bounds for the degree of nonnegativity certificates.

The bound

Lemma

Suppose $f \in \mathbb{R}_d[I_n]$*, vanishes on* T_t*. If* $d \leq t \leq n - d$*, then* f *is properly divisible by* $\ell = t - \sum x_i$*.*

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Theorem

Suppose $f \in \mathbb{R}_t[I_n]$ with $t \leq n/2$ is an S_n-invariant polynomial and f is properly divisible by $\ell = t - (x_1 + \cdots + x_n)$ to odd order. Then f is not d-rsos for $d \leq t$.

Quadratics

In particular we have:

Theorem

Let $k = \lfloor \frac{n}{2} \rfloor$ *and let* $f \in \mathbb{R}[I_n]$ *be given by*

$$f = (x_1 + \cdots + x_n - k)(x_1 + \cdots + x_n - k - 1).$$

Then f *is nonnegative on* C_n *but* f *is not* k*-rsos.*

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Then f is nonnegative on C_n but f is not k-rsos.

This shows our upper bound was tight.

Section 4

Applications

Globally nonnegative polynomials

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Corollary

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MaxCut

The maxcut problem over K_n can be reduced to

Binary polynomial formulation of MaxCut

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Note that p attains its maximum in C_n at T_k and T_{k+1} so

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A weaker version can now be proved.

Theorem

If $n = 2k + 1$, $(p_\omega)_{\max} = (p_\omega)^{k+1}_{\text{rsos}}$ for all weights or $(p_\omega)^{k+2}_{\text{rsos}}$ if we want positive multipliers.

Thank You