Higher derived brackets and homotopy algebras

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Abstract

We give a construction of homotopy algebras based on “higher derived brackets”. More precisely, the data include a Lie superalgebra with a projector on an Abelian subalgebra satisfying a certain axiom, and an odd element $\Delta$. Given this, we introduce an infinite sequence of higher brackets on the image of the projector, and explicitly calculate their Jacobians in terms of $\Delta^2$. This allows to control higher Jacobi identities in terms of the “order” of $\Delta$. Examples include Stasheff’s strongly homotopy Lie algebras and variants of homotopy Batalin–Vilkovisky algebras. There is a generalization with $\Delta$ replaced by an arbitrary odd derivation. We discuss applications and links with other constructions.

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1. Introduction

Strong homotopy Lie algebras (“strongly homotopy”, sh Lie algebras, $L_\infty$-algebras) were defined by Lada and Stasheff in [25] (see also [24]). According to Stasheff (private communication), this notion was “recognized” by him when algebraic structures such as string products of Zwiebach [37], and similar, started to appear in physical works. Before that, Schlessinger and Stasheff [31] realized that the notion of $L_\infty$-algebra was relevant to describing the higher order obstructions occurring in deformation theory, though this was not described in the paper [31]. Note also the work by Retakh [27]. The associative counterpart
of the $L_\infty$-algebras, Stasheff’s $A_\infty$-algebras became widely known much earlier. Currently, all kinds of homotopy algebras and structures related to them attract great attention. In part, this is due to their applications such as in Kontsevich’s proof of the existence of deformation quantization for any Poisson manifold. For an operadic approach to such algebras, see [26].

In this paper, we give a rather general algebraic construction that produces strong homotopy Lie algebras (and related algebras) from simple data. Namely, we consider a Lie superalgebra $L$ with a projector on an Abelian subalgebra obeying a “distributivity” condition (2). There are many examples of such projectors. Now, given this, an element $\Delta$ defines a sequence of $n$-ary brackets on the image of the projector $P$ as

$$\{a_1, \ldots, a_n\} := P[[\ldots[[\Delta, a_1], a_2], \ldots, a_n],$$

where $a_i$ are in the image of $P$. We call them higher derived brackets and we call $\Delta$ the generator for the derived brackets. We prove that for an odd $\Delta$, the $n$th Jacobiator of these derived brackets (i.e., the LHS of the $n$th Jacobi identity of the $L_\infty$-algebras) exactly equals the $n$th derived bracket for the element $\Delta^2$. Hence, if $\Delta^2 = 0$, our construction leads to strong homotopy Lie algebras. We can weaken the condition $\Delta^2 = 0$ still obtaining the Jacobi identities of higher orders. This naturally occurs in examples. Particularly interesting applications of this construction are to higher Poisson brackets and brackets generated by a differential operator, which give an important example of a (strong) “homotopy Batalin–Vilkovisky algebra”. Our construction as a particular case contains the well-known description of $L_\infty$-algebras in terms of homological vector fields. Though it is a generating element $\Delta$ that plays a key role in the main examples, it is also possible to give a similar construction of higher derived brackets taking as a starting point an arbitrary odd derivation $d : L \to L$; in particular, this allows to give a homotopy-theoretic interpretation of higher derived brackets.

In Section 2 we introduce the setup and recall the notion of $L_\infty$-algebras (in a form convenient for our purposes). In Section 3 we state and prove the main theorem. Sections 2 and 3 are purely algebraic and self-contained. In Section 4 we consider some examples of applications. In Section 5 we return to algebra, giving a sketch of the generalization of our construction for non-inner derivations and applying it to a homotopy-theoretic interpretation. Finally, in Section 6 we discuss related works, links with our results and directions for further study. (Among other things we explain the role of $P$ and the necessity of higher brackets, compared to a binary derived bracket as in [20].)

**Terminology and notation:** We work in the $\mathbb{Z}_2$-graded (super) context, e.g., a vector space means a ‘$\mathbb{Z}_2$-graded vector space’, etc. Tilde over a symbol denotes parity. (A parallel treatment for the $\mathbb{Z}$-graded context is possible.)

**2. Setup and preliminaries**

Let $L$ be a Lie superalgebra. Consider a linear projector $P \in \text{End } L$, $P^2 = P$, such that the image of $P$ is an Abelian subalgebra

$$[Pa, Pb] = 0$$

(1)
for all \( a, b \in L \). Let \( P \) also satisfy the following distributive law w.r.t. the commutator

\[
P[a, b] = P[Pa, b] + P[a, Pb].
\]  

(2)

This identity is a convenient way of expressing the requirement that the kernel of \( P \) is also a subalgebra in \( L \) (not necessarily Abelian). Consider an arbitrary odd element \( \Delta \in L \). Using these data, \( P \) and \( \Delta \), we shall introduce a sequence of \( n \)-ary brackets on the vector space \( P(L) \subseteq L \), the image of the projector \( P \), and check that upon certain conditions they will make it into a strongly homotopy Lie algebra. More precisely, we shall see how the corresponding identities are controlled by the properties of the element \( \Delta^2 = \frac{1}{2} [\Delta, \Delta] \) and arise step by step.

Let us give some examples of a projector \( P \).

**Example 2.1.** Let \( V = V_0 \oplus V_1 \) be a \( \mathbb{Z}_2 \)-graded vector space, which we also treat as a supermanifold. The origin 0 is a distinguished point. Take as \( L \) the superalgebra \( \text{Vect}(V) \) of all vector fields on the supermanifold \( V \) w.r.t. the usual Lie bracket. Let \( P \) take every vector field to its value at the origin considered as a vector field with constant coefficients. One can check that the map \( P : X \mapsto X(0) \) satisfies (2).

**Example 2.2.** Let \( A \) be a commutative associative algebra with a unit, and let \( L = \text{End} A \) (the space of all linear operators in \( A \)) with the usual commutator of operators as a bracket. The map \( P : \Delta \mapsto \Delta(1) \) maps every operator to an element of \( A \), which can be identified with an operator of left multiplication. The image of \( P \) is an Abelian subalgebra in \( \text{End} A \). Again, a direct check shows that \( P \) satisfies (2).

**Example 2.3.** Let \( M \) be a supermanifold, and \( T^* M \) its cotangent bundle. Take as \( L \) the Lie superalgebra \( C^\infty(T^* M) \) w.r.t. the canonical Poisson bracket. Define \( P \) as the pullback of functions on \( T^* M \) to \( M \). \( C^\infty(M) \) can be treated as a subspace of \( C^\infty(T^* M) \); in particular, it is an Abelian subalgebra. It is directly checked that \( P \) satisfies (2). In view of the relation between the commutator of operators and the Poisson bracket, this example can be seen as a ‘classical counterpart’ of Example 2.2.

Let us recall the definition of a strongly homotopy Lie algebra due to Stasheff. In a form convenient for our purposes it reads as follows.

**Definition 1.** A vector space \( V = V_0 \oplus V_1 \) endowed with a sequence of odd \( n \)-linear operations, \( n = 0, 1, 2, 3, \ldots \) (which we denote by braces), is a (strongly) homotopy Lie algebra or \( L_\infty \)-algebra if: (a) all operations are symmetric in the \( \mathbb{Z}_2 \)-graded sense

\[
\{a_1, \ldots, a_i, a_{i+1}, \ldots, a_n\} = (-1)^{\hat{a}_i \hat{a}_{i+1}} \{a_1, \ldots, a_{i+1}, a_i, \ldots, a_n\}
\]

(3)

and (b) the “generalized Jacobi identities”

\[
\sum_{k+l=n} \sum_{(k,l)\text{-shuffles}} (-1)^2 \{\{a_{\sigma(1)}, \ldots, a_{\sigma(k)}\}, a_{\sigma(k+1)}, \ldots, a_{\sigma(k+l)}\} = 0
\]

(4)

hold for all \( n = 0, 1, 2, \ldots \). Here \((-1)^2\) is the sign prescribed by the sign rule for a permutation of homogeneous elements \( a_1, \ldots, a_n \in V \).
Henceforth symmetric will mean \( \mathbb{Z}_2 \)-graded symmetric.

The notation is such that the parity of each operation "sits" at the opening bracket, which should be regarded as an odd symbol w.r.t. the sign rule. A 0-ary bracket is just a distinguished element \( \{\emptyset\} \) in \( L \). Recall that a \((k, l)\)-shuffle is a permutation of indices \( 1, 2, \ldots, k + l \) such that \( \sigma(1) < \cdots < \sigma(k) \) and \( \sigma(k + 1) < \cdots < \sigma(k + l) \). Below are the generalized Jacobi identities for \( n = 0, 1, 2, 3 \)

\[
\{\emptyset\} = 0, \quad (5)
\]

\[
\{\{a\}\} + \{\emptyset, a\} = 0, \quad (6)
\]

\[
\{\{a, b\}\} + \{\{a\}, b\} + (-1)^{\bar{a}\bar{b}}\{\{b\}, a\} + \{\emptyset, a, b\} = 0, \quad (7)
\]

\[
\{\{a, b, c\}\} + \{\{a, b\}, c\} + (-1)^{\bar{a}\bar{b}\bar{c}}\{\{b, c\}, a\} + \{\emptyset, a, b, c\} = 0. \quad (8)
\]

We shall call the \( L_\infty \)-algebras with \( \emptyset = 0 \), strict.

For strict \( L_\infty \)-algebras, the Jacobi identities start from \( n = 1 \), and in (4) the summation is over \( k \geq 0, \ l > 0 \). Identities (6)–(8) for strict \( L_\infty \)-algebras simplify to

\[
d^2 a = 0, \quad (9)
\]

\[
d\{a, b\} + \{da, b\} + (-1)^{\bar{a}\bar{b}}\{db, a\} = 0, \quad (10)
\]

\[
d\{a, b, c\} + \{\{a, b\}, c\} + (-1)^{\bar{a}\bar{b}\bar{c}}\{\{b, c\}, a\} + \{\emptyset, a, b, c\} = 0. \quad (11)
\]

if we denote the unary bracket as \( d := \{\_\} \). That is, \( d \) acts as a differential, it has the derivation property w.r.t. the binary bracket, and the usual Jacobi holds for the binary bracket with a homotopy correction. The identities with \( n > 3 \) impose extra relations for this homotopy and all the higher homotopies (hence ‘strongly’ in the name).

As the operations \( \{a_1, \ldots, a_n\} \) are multilinear and symmetric, they are completely determined by the values on coinciding even arguments: \( \{\xi, \ldots, \xi\} \) where \( \xi \) is an even element of \( V \) (to this end, extension of scalars by odd constants should be allowed). A generating function for these operations can be conveniently written as a (formal) odd vector field on the vector space \( V \) considered as a supermanifold

\[
Q = Q^i (\xi) \frac{\partial}{\partial \xi^i} := \sum_{n \geq 0} \frac{1}{n!} \{\xi, \ldots, \xi\}. \quad (12)
\]

The elements of \( V \) are identified with (constant) vector fields as \( u = u^i e_i \leftrightarrow u^i \partial_j \). If we denote the \( n \)th Jacobiator, i.e., the LHS of (4), by \( J^n(a_1, \ldots, a_n) \), it is clear that \( J^n \) also give multilinear symmetric operations on \( V \). Hence they are, too, defined by their values on
equal even arguments. The expression simplifies greatly, and we have
\[ J^n(\xi_1, \ldots, \xi_n) = n \sum_{l=0}^{n} \frac{n!}{l!(n-l)!} \{(\xi_1, \ldots, \xi_l), \xi_{l+1}, \ldots, \xi_n\} \] (13)
for an even \( \xi \). Abbreviating \( J^n(\xi_1, \ldots, \xi_n) \) to \( J^n(\xi) \) we can write a generating function as
\[ J := \sum_{n \geq 0} \frac{1}{n!} J^n(\xi), \] (14)
which is an even (formal) vector field on the supermanifold \( V \). One can directly see that
\[ J = Q^2 = \frac{1}{2}[Q, Q]. \] Hence all the Jacobi identities can be compactly written as \( Q^2 = 0 \).

Note that for strict \( L_\infty \)-algebras the vector fields \( Q \) and \( J = Q^2 \) vanish at the origin.

Remark 2.1. There is a difference between the sign conventions of our Definition 1 and the ‘standard’ definitions of the \( L_\infty \)-algebras as in [25,24]. It comes from two sources. First, there is a choice between the ‘graded’ (= \( \mathbb{Z} \)-graded) and ‘super’ viewpoints. Second, in supermathematics one can choose between ‘symmetric’ and ‘antisymmetric’ constructions using the parity shift. In [25,24] all vector spaces are \( \mathbb{Z} \)-graded, but not ‘super’, and brackets are antisymmetric in the graded sense, i.e., involving the usual signs of permutations together with the ‘Koszul signs’ coming from the \( \mathbb{Z} \)-grading. We prefer to work with the ‘super’ conventions where all the signs come from the \( \mathbb{Z}_2 \)-grading (but not from any extra \( \mathbb{Z} \)-grading be it present), and our brackets are (super) symmetric. This has an advantage that it allows to use geometric language and certain signs are simplified (e.g., the signs of permutations do not enter). On the other hand, the definitions in [25,24] include directly the ordinary Lie algebras as a particular case. A passage from [25,24] to our conventions consists in introducing a \( \mathbb{Z}_2 \)-grading (parity) as the degree mod 2 and applying the parity shift. Notice that it reverses the parities of brackets with even numbers of arguments and turns antisymmetric operations into symmetric. More precisely, let \( \Pi \) be the parity reversion functor. Suppose \( V = \Pi g \). If we relate operations in \( V \) and \( g \) by the equality
\[ \Pi[x_1, \ldots, x_n] = (\Pi x_1, \ldots, \Pi x_n) (-1)^\varepsilon, \]
where \( x_i \in g \), then (assuming that all brackets in \( V \) are odd), the brackets in \( g \) with an even number of arguments will be even and with odd will be odd; the antisymmetry of brackets in \( g \) is equivalent to the symmetry of brackets in \( V \); the Jacobi identities in the form of [25,24] for the brackets in \( g \), extending the ordinary Jacobi identity for Lie algebras, are equivalent to the Jacobi identities in the form (4) for the brackets in \( V \). One might prefer to call such a \( V = \Pi g \), an ‘\( L_\infty \)-antialgebra’. However, we shall stick to Definition 1 throughout this paper. Notice that our conventions are close to those in [37].

Remark 2.2. In almost all standard approaches to \( L_\infty \)-algebras there is no 0-ary bracket or, rather, it is assumed that the corresponding element \( \Phi = \{\emptyset\} \) is zero. (Except in [37] and some other physical works; the algebras with a non-zero \( \Phi = \{\emptyset\} \) are called sometimes ‘weak’ or ‘with background’.) Hence the standard \( L_\infty \)-algebras are always ‘strict’ in our sense. We have allowed for a 0-ary operation because \( \Phi \neq 0 \) does occur naturally in some of our examples, and even where it does not, including it sometimes simplifies the exposition.
3. Main theorem

Let us return to the setting described above, i.e., a Lie superalgebra $L$ with a projector $P$ on an Abelian subalgebra, and an element $\Delta \in L$. Forget for a moment about restrictions on $\Delta$.

**Definition 2.** For an arbitrary element $\Delta \in L$, even or odd, we call the $n$th derived bracket of $\Delta$ the following operation on the subspace $V := P(L) \subset L$

$$\{a_1, \ldots, a_n\}_\Delta := P[\ldots[[\Delta, a_1], a_2], \ldots, a_n],$$  \hspace{1cm} (15)

where $a_i \in V$. Here $n = 0, 1, 2, 3, \ldots$.

We get a set of $n$-ary operations (15) on the space $V$. Clearly, they are multilinear and of the same parity as $\Delta$. (For $n = 0$, we get $\{\} = P(\{\})$. Notice that they are always symmetric. Indeed, for the interchange of $a_1$ and $a_2$, since $[[\Delta, a_1], a_2] = 0$, we have $\{a_1, a_2\} = (\Delta, a_1)[[\Delta, a_2], a_1] = (\Delta, a_2)[[\Delta, a_1], a_1]$, where $\varepsilon = \hat{\Delta}$. Hence

$$\{a_1, a_2, \ldots, a_n\}_\Delta = (-1)^{\hat{\Delta}_1 \hat{\Delta}_2} \{a_2, a_1, \ldots, a_n\}_\Delta.$$

Similarly for other adjacent arguments. For the coinciding even arguments of the $n$th derived bracket we have

$$\{\xi, \ldots, \xi\}_\Delta = P(\text{ad} \xi)^n \Delta$$  \hspace{1cm} (16)

(which is reminiscent of the $n$th derivative of an $f(x)$ at a point $x_0$).

In the sequel we shall be particularly interested in the derived brackets of an odd element $\Delta$ and of its square $\Delta^2 = \frac{1}{2}[[\Delta, \Delta]]$.

Let $\Delta \in L$ be odd. Consider the Jacobians $J^n_\Delta(\tilde{\Delta})$ for the derived brackets of $\Delta$. From (13) and (16) we get

$$J^n_\Delta(\tilde{\Delta}) = (-1)^n \sum_{l=0}^n \frac{n!}{l!(n-l)!} P(\text{ad} \tilde{\Delta})^l \Delta, P(\text{ad} \tilde{\Delta})^{n-l} \Delta]$$

$$= (-1)^n \sum_{l=0}^n \frac{n!}{l!(n-l)!} P[\text{ad} \tilde{\Delta}]^l \Delta, P(\text{ad} \tilde{\Delta})^{n-l} \Delta],$$  \hspace{1cm} (17)

where to obtain the second equality we used the Leibniz formula for $(\text{ad} \tilde{\Delta})^l$ w.r.t. the Lie bracket in $L$ and the vanishing of the commutators between elements of $V \subset L$.

**Theorem 1.** Suppose $P$ satisfies (1) and (2). Let $\Delta$ be an arbitrary odd element. Then the $n$th Jacobiator $J^n_\Delta$ for the derived brackets of $\Delta$ is exactly the $n$th derived bracket of $\Delta^2$

$$J^n_\Delta(a_1, \ldots, a_n) = \{a_1, \ldots, a_n\}_{\Delta^2}. \hspace{1cm} (18)$$
\textbf{Proof.} We shall prove the required identity for the coinciding even arguments

\[ J^\mathbf{n}_\Delta(\xi) = \{\xi, \ldots, \xi\}_\Delta. \quad (19) \]

Indeed, for the LHS we can apply formula (17). Let us analyze the cases of \( n \) odd and \( n \) even separately. Suppose \( n = 2m + 1 \). Then we have

\[
-J^2m+1_\Delta(\xi) = P[\Delta, P(\ad \xi)^{2m+1} \Delta] + \frac{(2m + 1)!}{1!(2m)!} P[\ad \xi, \Delta, P(\ad \xi)^{2m} \Delta]
+ \cdots + \frac{(2m + 1)!}{m!(m + 1)!} P[\ad \xi]^m \Delta, P(\ad \xi)^{m+1} \Delta]
+ \frac{(2m + 1)!}{(m + 1)!m!} P[\ad \xi]^{m+1} \Delta, P(\ad \xi)^m \Delta]
+ \cdots + \frac{(2m + 1)!}{(2m)!1!} P[\ad \xi]^2m \Delta, P(\ad \xi, \Delta]
+ P[(\ad \xi)^{2m+1} \Delta, P(\Delta)].
\]

The terms corresponding to \( l \) and \( 2m + 1 - l \), where \( l = 0, 1, \ldots, m \) can be grouped in pairs, and to each of the pairs we can apply the distributive law (2). Thus we get after taking \( P \) out

\[
J^2m+1_\Delta(\xi) = -P \left( \sum_{l=0}^{m} \frac{(2m + 1)!}{l!(2m + 1 - l)!} [(\ad \xi)^l \Delta, (\ad \xi)^{2m+1-l} \Delta] \right)
= -\frac{1}{2} P \left( \sum_{l=0}^{m+1} \frac{(2m + 1)!}{l!(2m + 1 - l)!} [(\ad \xi)^l \Delta, (\ad \xi)^{2m+1-l} \Delta] \right)
= -\frac{1}{2} P(\ad \xi)^{2m+1} [\Delta, \Delta] = P(\ad \xi)^{2m+1} \Delta^2.
\]

Here we used the Leibniz identity for \( (\ad \xi)^{2m+1} \) w.r.t. the commutator in \( L \). Now suppose \( n = 2m > 0 \). We have

\[
+J^2m_\Delta(\xi) = P[\Delta, P(\ad \xi)^{2m} \Delta] + \frac{(2m)!}{1!(2m - 1)!} P[\ad \xi, \Delta, P(\ad \xi)^{2m-1} \Delta]
+ \cdots + \frac{(2m)!}{(m - 1)!(m + 1)!} P[\ad \xi]^{m-1} \Delta, P(\ad \xi)^{m+1} \Delta]
+ \frac{(2m)!}{m!m!} P[\ad \xi]^m \Delta, P(\ad \xi)^m \Delta]
+ \frac{(2m)!}{(m + 1)!(m - 1)!} P[\ad \xi]^{m+1} \Delta, P(\ad \xi)^{m-1} \Delta]
+ \cdots + \frac{(2m)!}{(2m-1)!1!} P[(\ad \xi)^{2m-1} \Delta, P(\ad \xi, \Delta]
+ \frac{(2m)!}{(2m)!0!} P[(\ad \xi)^{2m} \Delta, P(\Delta)].
\]

All terms except for the term with \( l = m \) can be grouped in pairs and transformed as above. To the term corresponding to \( l = m \) we can apply the identity \( P[a, a] = 2P[Pa, a] \), which
follows from the distributive law (2), valid for any odd \( a \in L \). Hence we get, similarly to the above

\[
J^m_\Delta(\zeta) = P \left( \sum_{l=0}^{m-1} \frac{(2m)!}{l!(2m-l)!} [(\text{ad } \zeta)^l \Delta, (\text{ad } \zeta)^{2m-l} \Delta] \right)
+ \frac{1}{2} \frac{(2m)!}{m!m!} P[(\text{ad } \zeta)^m \Delta, (\text{ad } \zeta)^m \Delta]
= \frac{1}{2} P \left( \sum_{l=0}^{2m} \frac{(2m)!}{l!(2m-l)!} [(\text{ad } \zeta)^l \Delta, (\text{ad } \zeta)^{2m-l} \Delta] \right)
= \frac{1}{2} P(\text{ad } \zeta)^{2m} [\Delta, \Delta] = P(\text{ad } \zeta)^{2m} \Delta^2.
\]

For completeness, notice that for \( n = 0 \) we have \( J^0_\Delta = \{\emptyset\}_\Delta = P[\Delta, P(\Delta)] = \frac{1}{2} P[\Delta, \Delta] = P(\Delta^2) = \{\emptyset\}_\Delta^2 \). We conclude that in all cases

\[
J^n_\Delta(\zeta) = P(-\text{ad } \zeta)^n \Delta^2 = \{\zeta, \ldots, \zeta\}_n \Delta^2, \quad (20)
\]

as claimed. \( \square \)

**Corollary 1.** In the setup of Theorem 1, if \( \Delta^2 = 0 \), the derived brackets of \( \Delta \) make \( V \) an \( L_\infty \)-algebra.

This allows a generalization, which naturally comes up in examples.

**Definition 3.** For any element \( \Delta \in L \) we define the number \( r \) to be the order of \( \Delta \) w.r.t. a subalgebra \( V \subset L \) if all \( (r+1) \)-fold commutators \( [\cdots [\Delta, a_1], \ldots, a_{r+1}] \) with arbitrary elements of \( V \) identically vanish. Notation: \( \text{ord}_V \Delta \).

This is a filtration in \( L \).

**Corollary 2.** In the setup of Theorem 1, if \( \text{ord}_V \Delta^2 \leq r \), then the derived brackets of \( \Delta \) satisfy the Jacobi identities of orders \( n > r \).

We call the algebras given by Corollary 2, \( L_\infty \)-algebras of order \( > r \).

Note that any higher Jacobi identity includes all \( n \)-brackets with \( n = 0, 1, \ldots \). As above, we can speak about strict \( L_\infty \)-algebras of order \( > r \) if the 0-bracket \( \Phi \) vanishes. A natural question is, when one can split the element \( \Phi = P(\Delta) \) from the Jacobi identities of orders \( n \geq 1 \) and simply drop the 0-ary bracket from consideration. This happens if \( \Phi \) is an annihilator of all \( n \)-brackets, \( n = 2, 3, \ldots \). Besides an evident case \( \Phi = P(\Delta) = 0 \), a sufficient condition is \( P[\Delta, P(\Delta)] = [\Delta, P(\Delta)] \). See examples in the next section.
4. Applications

In this section, we consider some examples of applications of Theorem 1 and Corollaries 1, 2.

Example 4.1. Consider a vector space \( V \), with the algebra \( L = \text{Vect}(V) \) and the projector \( P \) as in Example 2.1. Take as \( \Omega \) an arbitrary odd vector field \( Q \in \text{Vect}(V) \)

\[
Q = Q^k(\xi) \frac{\partial}{\partial \xi^k} = \left( Q_0^k + \xi^i Q_i^k + \frac{1}{2} \xi^i \xi^j Q_{ij}^k + \frac{1}{3!} \xi^i \xi^j \xi^l Q_{ijl}^k + \cdots \right) \frac{\partial}{\partial \xi^k}.
\]

The derived brackets of \( Q \)

\[
\{a_1, \ldots, a_n\}_Q = \ldots [[Q, a_1], a_2], \ldots, a_n](0),
\]

where \( a_i \in V \) are identified with the corresponding constant vector fields, are given by the coefficients of the Maclaurin expansion:

\[
Q_0^k = Q_0^k e_k,
\]

\[
d e_i := \{e_i\} = (-1)^{\tilde{i} + 1} Q_i^k e_k, \quad \{e_i, e_j\} = (-1)^{\tilde{i} + \tilde{j}} Q_{ij}^k e_k,
\]

\[
\{e_i, e_j, e_l\} = (-1)^{\tilde{i} + \tilde{j} + \tilde{l} + 1} Q_{ijl}^k e_k, \ldots ,
\]

for the basis \( e_i \). Here we denoted \( \tilde{i} = \tilde{\epsilon}_i \). These are precisely the brackets on \( V \) for which the vector field \( Q \) (or, rather, its Maclaurin series) is the generating function (12). Hence for \( Q^2 = 0 \) we recover the 1–1-correspondence between \( L_\infty \)-algebras and homological vector fields. Moreover, we see that it is given by the explicit formula (21). Set \( Q_0 = 0 \). Then the algebra is strict. For vector fields on \( V \), the order w.r.t. the subalgebra of constant vector fields \( V \subset \text{Vect}(V) \) is the degree in the variables \( \xi^i \) (as a filtration). We conclude that strict \( L_\infty \)-algebras of order \( > r \) are in a 1–1-correspondence with odd vector fields \( Q \) vanishing at the origin with the square \( Q^2 \) of degree \( \leq r \) in coordinates.

Example 4.2. For a (super)manifold \( M \), consider \( C^\infty(M) \subset C^\infty(T^* M) \) as in Example 2.3. The projector is the pullback. Any odd Hamiltonian \( S \in C^\infty(T^* M) \) defines a sequence of higher Schouten (\( = \) odd Poisson) brackets in \( C^\infty(M) \) by the formula

\[
\{ f_1, \ldots, f_n \}_S := \ldots (((S, f_1), f_2), \ldots, f_n)|_{p=0}
\]

(the parentheses stand for the canonical Poisson bracket on \( T^* M \)). Here \( f_1, \ldots, f_n \in C^\infty(M) \). They satisfy the Jacobi identities of all orders if \( (S, S) = 0 \). Note that a Hamiltonian has a finite order w.r.t. the subalgebra \( C^\infty(M) \) if it is polynomial in \( p_a \), and the order is the respective degree. The Jacobi identities can be obtained one by one by putting restrictions on the order of \( (S, S) \). If

\[
S = S(x, p) = S_0(x) + S^a(x) p_a + \frac{1}{2} S^{ab}(x) p_b p_a + \frac{1}{3!} S^{abc}(x) p_c p_b p_a + \cdots ,
\]

then
\[
\{\emptyset\} = S_0, \quad \delta f := \{f\} = S^a \partial_a f, \quad \{f, g\} = (-1)^{\delta a} S^{ab} \partial_b f \partial_a g,
\]
\[
\{f, g, h\} = (-1)^{\delta (a + b) + \delta c} S^{abc} \partial_c f \partial_b g \partial_a h, \ldots.
\]
If \((S, S)\) is of degree \(\leq r\) in \(p_a\), then the brackets satisfy the Jacobi identities of orders \(\geq r + 1\). In this example each of the higher Schouten brackets is a multi-derivation, i.e., satisfies the Leibniz rule w.r.t. the usual product, in each argument. Hence the algebras that we obtain are particular homotopy analogs of odd Poisson (=Schouten, Gerstenhaber) algebras. The ‘strict’ case is when \(S_{|p=0} = 0\).

**Example 4.3.** Similarly to the above, take as \(L\) the algebra of multivector fields \(C^\infty(\Pi T^* M)\) with the canonical Schouten bracket. Here we have to change parity to obtain a Lie super-algebra. The rest goes as in Example 4.2. Any even multivector field \(P \in C^\infty(\Pi T^* M)\) provides a sequence of higher Poisson brackets in \(C^\infty(M)\):
\[
\{f_1, \ldots, f_n\}_P := \llbracket \ldots \llbracket P, f_1 \rrbracket, f_2 \rrbracket, \ldots, f_n \rrbracket_{x^*=0}.
\]
The brackets of odd orders are odd, the brackets of even orders are even. We have
\[
\{f_1, \ldots, f_n\}_P = P^{a_1 \ldots a_n}(x) \partial_{a_n} f_n \ldots \partial_{a_1} f_1
\]
for even functions (for arbitrary functions the formula follows by linearity, using multiplication by odd constants), where
\[
P = P(x, x^*) = P_0(x) + P^a(x) x^a + \frac{1}{2} P^{ab}(x) x^a x^b + \frac{1}{3!} P^{abc}(x) x^a x^b x^c + \ldots,
\]
with the full set of the Jacobi identities being equivalent to \(\llbracket P, P \rrbracket = 0\). Again, there is a possibility of getting the Jacobi identities step by step by putting restrictions on the degree of \(\llbracket P, P \rrbracket\). As in Example 4.2, each of the higher brackets strictly satisfies the Leibniz rule w.r.t. the product of functions.

Examples 4.3 and 4.2 generalize classical Poisson and Schouten (=odd Poisson) structures, as Example 4.1 generalizes classical Lie algebras. Indeed, for a bivector field \(P = \frac{1}{2} P^{ab} x^b x^a\) or an odd Hamiltonian quadratic in the momenta \(S = \frac{1}{2} S^{ab} p_b p_a\), the binary derived bracket is an ordinary Poisson or Schouten bracket, respectively, and all other brackets vanish. (Similarly, after the shift of parity the bracket in a Lie algebra is the binary derived bracket for a homological vector field \(Q = \frac{1}{2} \xi^i \xi^j Q_{ij} \partial_i \) that is quadratic in coordinates.) A mechanism for the arising of higher brackets can be to take a quadratic Hamiltonian or a bivector field generating an ordinary Schouten or Poisson bracket, and apply to it a canonical transformation that fixes the zero section but not the bundle structure.

Note in both examples the possibility of obtaining higher odd or even Poisson brackets from a non-polynomial Hamiltonian or multivector field (the latter is possible only in the super case). It is the Taylor expansion around the zero section \(M \subset T^* M\) or \(M \subset \Pi T^* M\) that counts.
Example 4.4. Consider a commutative associative algebra with a unit \( A \), e.g., an algebra of smooth functions \( C^\infty(M) \). Let \( L \) be the algebra of all linear operators in \( A \) w.r.t. the commutator, and \( V = A \) considered as an Abelian subalgebra in \( L \). Let \( P : \text{End} \ A \to \text{End} \ A \) be the evaluation at 1, as in Example 2.2. Let \( \Delta \) be an arbitrary odd operator in \( A \). The derived brackets of \( \Delta \)

\[
\{f_1, \ldots, f_n\}_\Delta := \ldots[[\Delta, f_1], f_2], \ldots, f_n](1),
\]

will be, respectively

\[
\{\emptyset\} = \Phi = \Delta 1,
\]

\[
\{f\} = \Delta' f = \Delta f - \Delta 1 \cdot f,
\]

\[
\{f, g\}_\Delta = \Delta(fg) - \Delta f \cdot g - (-1)^{\bar{f}} f \cdot \Delta g + \Delta 1 \cdot fg,
\]

\[
\{f, g, h\}_\Delta = \Delta(fgh) - \Delta(fg) \cdot h - (-1)^{\bar{g}} \Delta(fh) \cdot g
\]

\[
\quad - (-1)^{\bar{f}+\bar{h}} \Delta(gh) \cdot f + \Delta f \cdot gh
\]

\[
\quad + (-1)^{\bar{f}+\bar{h}} \Delta g \cdot fh + (-1)^{\bar{g}+\bar{h}} \Delta h \cdot fg - \Delta 1 \cdot fgh
\]

One can check that these brackets satisfy the following identity w.r.t. the product of functions:

\[
\{f_1, \ldots, f_{n-1}, gh\}_\Delta = \{f_1, \ldots, f_{n-1}, g\}_\Delta h + (-1)^{\bar{g}\bar{h}} \{f_1, \ldots, f_{n-1}, h\}_\Delta g + \{f_1, \ldots, f_{n-1}, g, h\}_\Delta,
\]

i.e., the \((n+1)\)th bracket arises as the failure of the Leibniz rule for the \(n\)th bracket. If \( \Delta \) is a differential operator of order \( s \), then the \((s + 1)\)th bracket and all higher brackets identically vanish, and the \(s\)th bracket is a (symmetric) multi-derivation of the algebra \( A \). (It is nothing but the polarization of the principal symbol of \( \Delta \).) The usual order of a differential operator is exactly the ‘order’ w.r.t. the subalgebra \( A \). The \( 4 \)th bracket with \( 1 \leq k \leq s \) is in this case a differential operator of order \( s - k + 1 \) in each argument. One can view these brackets as consecutive “polarizations” of the operator \( \Delta \). It is instructive to write them down explicitly for a particular operator \( \Delta \) in a differential-geometric setting (see below). As follows from Theorem 1, if \( \Delta^2 = 0 \), then the derived brackets of \( \Delta \) satisfy the Jacobi identities of all orders; otherwise, by requiring \( \text{ord} \Delta^2 \leq r \) we obtain the Jacobi identities of orders \( r + 1 \) and higher.

Remark 4.1. That brackets (22) give an \( L_\infty \)-algebra if \( \Delta^2 = 0 \) was for the first time proved in [9], by rather hard calculations.

The \( n \)-brackets (22) with \( n \geq 1 \) will not change if we replace \( \Delta \) by \( \Delta' = \Delta - \Delta 1 \). Let \( J^n_\Delta \) denote the \( n \)th Jacobiator with \( \Phi \) dropped, and \( J^n_\Delta \) stands for the full Jacobiator, \( n > 0 \). Then \( J^n_\Delta = J^n_\Delta' \). Applying Theorem 1, we identify \( J^n_\Delta \) and \( J^n_\Delta' \) with the \( n \)-brackets generated by \( \Delta^2 \) and \( \Delta'^2 \), respectively. Since \( \Delta'^2 = \Delta^2 - [\Delta, \Delta 1] \), by comparing the orders we conclude
that $J_{\Delta}^{n} = J_{\Delta}^{n}$ for all $n = s, s + 1, \ldots, 2s - 1$ if ord $\Delta \leq s$. Hence $\Phi = \Delta 1$ can be dropped from the $n$th Jacobi identity for the brackets generated by $\Delta$ exactly for these numbers $n$.

The construction in Example 4.4 generalizes the interpretation of a classical odd Poisson bracket as the derived bracket of a ‘generating operator’ of second order (= an odd Laplacian, a ‘BV-operator’). This approach was particularly useful for the analysis of second order differential operators in [16, 17] (see also [18]).

**Example 4.5 ([16, 17])**. If $\Delta$ is an odd second-order differential operator in $C^\infty (M)$, in local coordinates

$$\Delta = R(x) + T^a(x) \partial_a + \frac{1}{2} S^{ab}(x) \partial_b \partial_a,$$

then we get

$$\Phi = \Delta 1 = R,$$

$$\{f\}_{\Delta} = \Delta' = T^a \partial_a f + \frac{1}{2} S^{ab} \partial_b \partial_a f,$$

$$\{f, g\}_{\Delta} = (-1)^{f \partial a} S^{ab} \partial_b f \partial_a g.$$

All the higher brackets vanish. Automatically ord $\Delta^2 \leq 3$. If ord $\Delta^2 \leq 2$, then $\{f, g\}_{\Delta}$ satisfies the usual Jacobi identity, making $C^\infty (M)$ into an odd Poisson algebra. If ord $\Delta^2 \leq 1$, then $\Delta'$ is a derivation of the bracket. Finally, if ord $\Delta^2 \leq 0$ and $\Delta 1 = 0$, then $\Delta = \Delta'$ is a differential; the resulting algebraic structure is known as a Batalin–Vilkovisky algebra. (Note that $\Delta 1$ does not affect the Jacobi identities with $n = 2, 3$.)

**Example 4.6.** For an odd third-order differential operator in $C^\infty (M)$, in local coordinates

$$\Delta = R(x) + T^a(x) \partial_a + \frac{1}{2} U^{ab}(x) \partial_b \partial_a + \frac{1}{3!} S^{abc}(x) \partial_c \partial_b \partial_a,$$

we get

$$\Phi = \Delta 1 = R,$$

$$\{f\}_{\Delta} = \Delta' = T^a \partial_a f + \frac{1}{2} U^{ab} \partial_b \partial_a f + \frac{1}{3!} S^{abc} \partial_c \partial_b \partial_a f,$$

$$\{f, g\}_{\Delta} = (-1)^{f \partial a} \left( U^{ab} \partial_b f \partial_a g + \frac{1}{2} S^{abc} (-1)^{f \partial b} \partial_c f \partial_b \partial_a g + \partial_c \partial_b f \partial_a g \right),$$

$$\{f, g, h\}_{\Delta} = (-1)^{f (\partial a + \partial b) + \partial a} S^{abc} \partial_c f \partial_b g \partial_a h,$$

and all the higher brackets vanish. Automatically ord $\Delta^2 \leq 5$. Not affected by $\Delta 1$ are the Jacobi identities with $n = 3, 4, 5$. If ord $\Delta^2 \leq 4$, then there holds the fifth-order Jacobi identity

$$\sum_{\text{shuffles}} \pm \{f, g, h\}_{\Delta}, e, k}_{\Delta} = 0.$$
It involves only the ternary bracket. If \( \text{ord} \Delta^2 \leq 3 \), then also holds the fourth-order Jacobi identity
\[
\sum_{\text{shuffles}} \pm \{(f, g, h)_{\Delta}, e\}_{\Delta} + \sum_{\text{shuffles}} \pm \{(f, g)_{\Delta}, h, e\}_{\Delta} = 0.
\]
If \( \text{ord} \Delta^2 \leq 2 \), then in addition holds the third-order Jacobi identity
\[
\sum_{\text{cycle}} \pm \{\{f, g\}_{\Delta}, h\}_{\Delta} \pm \Delta'\{f, g, h\}_{\Delta} \\
\pm \{f, \Delta'g, h\}_{\Delta} \pm \{f, g, \Delta'h\}_{\Delta} = 0.
\]
If \( \text{ord} \Delta^2 \leq 1 \), we get the second-order Jacobi identity involving \( \Delta^1 = R \), which now cannot be ignored
\[
\Delta'\{f, g\}_{\Delta} \pm \{\Delta'f, g\}_{\Delta} \pm \{f, \Delta'g\}_{\Delta} + \{\Delta^1, f, g\} = 0.
\]
Finally, if \( \text{ord} \Delta^2 \leq 0 \), we arrive at the first-order Jacobi identity in the form \((\Delta')^2 f + \{\Delta^1, f\} = 0\). We have to impose \( \Delta^1 = 0 \) to get strictness back.

**Remark 4.2.** The algebraic structure consisting of all higher derived brackets of an odd differential operator of order \( n \) and the usual multiplication, should be considered an example of a *homotopy Batalin–Vilkovisky algebra* (see [34,23,33] and a discussion in Section 6).

The behavior of the brackets in Examples 4.2 and 4.4 w.r.t. the multiplication, at the first glance seems very different. However, the identities satisfied by the algebras obtained in Example 4.2 can be seen as the “classical limit” of the identities for the algebras obtained in Example 4.4. Indeed, if we redefine the brackets in Example 4.4 by inserting the “Planck’s constant” \( \hbar \), as
\[
\{f_1, \ldots, f_n\}_{\Delta} := (-i\hbar)^{-n}[[[\Delta, f_1], f_2], \ldots, f_n](1),
\]
then they will satisfy the same Jacobi-type identities as before, but the “Leibniz identity” will now read
\[
\{f_1, \ldots, f_{n-1}, g, h\}_{\Delta} = \{f_1, \ldots, f_{n-1}, g\}_{\Delta}h + (-1)^{\hat{g}\hbar}\{f_1, \ldots, f_{n-1}, h\}_{\Delta}g \\
+ (-i\hbar)\{f_1, \ldots, f_{n-1}, g, h\}_{\Delta},
\]
which clearly becomes the strict derivation property when \( \hbar \to 0 \).

5. Case of non-inner derivations

Higher derived brackets generated by an element \( \Delta \) naturally arise in applications, as we saw it in the previous section. However, from theoretical reasons and from the viewpoint of further generalizations it seems natural to look also into a possibility to obtain a similar construction from an arbitrary derivation of the superalgebra \( L \) rather than inner derivations given by \( \Delta \in L \). It is indeed possible and in particular allows to look at higher
derived brackets from yet another angle. Here we shall briefly outline the construction and statements, leaving a more detailed exposition for another occasion.

As above, let \(L\) be a Lie superalgebra and \(P\) a projector satisfying the identities \([Pa, Pb] = 0\) and

\[P[a, b] = P[Pa, b] + P[a, Pb]\]

for all \(a, b \in L\). Recall that it means that both subspaces \(V = \text{Im} \ P\) and \(K = \text{Ker} \ P\) are subalgebras and \(V\) is Abelian. Consider an arbitrary, even or odd, derivation \(d\) of the Lie superalgebra \(L\). Let us assume that the kernel \(K\) of \(P\) is closed under \(d\); this is equivalent to the identity

\[PdP = Pd.\]  
(Note that we do not assume the image of \(P\), i.e., the subspace \(V\), to be closed under \(d\).)

**Definition 4.** The \(n\)th derived bracket of \(d\) is the following operation on the subspace \(V \subset L\)

\[\{a_1, \ldots, a_n\}_d := P[\ldots [da_1, a_2], \ldots, a_n],\]

where \(a_i \in V\). Here \(n = 1, 2, 3, \ldots\).

**Remark.** If \(V\) happens to be closed under \(d\), then all the \(n\)-brackets (24) with \(n > 1\), will vanish. So it is the non-commutativity of \(d\) with \(P\) that is the source of higher derived brackets.

Brackets (24) are even or odd depending on the parity of \(d\). Note that there is no 0-ary bracket, differently from the construction based on \(P\). Exactly as above follows (from the derivation property of \(d\), the Jacobi identity in \(L\) and the condition that the subalgebra \(V \subset L\) is Abelian) that all higher brackets (24) are symmetric in the \(\mathbb{Z}_2\)-graded sense.

**Theorem 2.** Suppose \(d\) is an odd derivation. Then the \(n\)th Jacobiator of the derived brackets of \(d\) is exactly the \(n\)th derived bracket of \(d^2\):

\[J^n_d(a_1, \ldots, a_n) = \{a_1, \ldots, a_n\}_{d^2}.\]  
Here \(n = 1, 2, 3, \ldots\). (In the formula for the Jacobiator the 0th bracket should be set to zero.)

In particular, if \(d^2 = 0\), the higher derived brackets of \(d\) make the subspace \(V\) a strict \(L_\infty\)-algebra. Clearly, it is also possible to weaken the condition \(d^2 = 0\) by considering instead of it a filtration by an ‘order’ of operators w.r.t. the subspace \(V\), as we did above for \(\Delta\).

Theorem 2 is a generalization of Theorem 1 if \(P(\Delta) = 0\). As we have seen it in the examples, this not always the case, so better to consider these two statements as independent, though closely related.
The construction of higher derived brackets from an arbitrary derivation makes it possible to give for them a nice homotopy-theoretic interpretation, as follows.\textsuperscript{1} Let $d$ be an odd derivation of the superalgebra Lie $L$ such that $d^2 = 0$. So $L$ together with $d$ is a differential Lie superalgebra. Consider the subalgebra $K = \text{Ker } P$ in $L$. It is a differential subalgebra. Consider the inclusion map $i: K \to L$. Forget for a moment about the algebra structure and consider it just as an inclusion of complexes. (For our purposes, a complex is a $\mathbb{Z}_2$-graded vector space with an odd endomorphism of square zero.) As topologists know, “every map can be made a fibration”, by applying a cocomylinder construction. The fiber of this fibration is known as a ‘homotopy fiber’ of the original map. What is a homotopy fiber for the inclusion $i: K \to L$? The claim is, it is the space $\Pi V$. Moreover, the higher derived brackets will make it a homotopy fiber in the category of $L_\infty$-algebras. More precisely, the following statements hold.

Let $i: K \to L$ be an arbitrary inclusion of complexes such that there is given a complementary subspace $V \subset L$ for $K$, so that $L = K \oplus V$. ($V$ is not necessarily a subcomplex.) Let $P$ be the projector onto $V$ parallel to $K$. The space $V$ becomes a complex with the differential $Pd$. Introduce into $L \oplus \Pi V$ an operator $D$ as follows

$$D(x, \Pi a) := (dx, -\Pi P(x + da))$$

for $x \in L$, $a \in V$. Then $D^2 = 0$ (check!). Consider the maps $j: K \to L \oplus \Pi V$ and $p: L \oplus \Pi V \to L$, where $j: x \mapsto (x, 0)$, $p: (x, \Pi a) \mapsto x$.

**Lemma 1.** The diagram

$$
\begin{array}{ccc}
K & \xrightarrow{i} & L \\
\downarrow{j} & & \downarrow{p} \\
L \oplus \Pi V & & 
\end{array}
$$

is a cocomylinder diagram in the category of complexes, i.e., the maps $j$ and $p$ are chain maps, $i = p \circ j$, the map $j: K \to L \oplus \Pi V$ is a monomorphism (‘cofibration’) and a quasi-isomorphism (‘weak homotopy equivalence’), and the map $p: L \oplus \Pi V \to L$ is an epimorphism (‘fibration’).

(A quasi-inverse for $j$ is $q: (x, \Pi a) \mapsto (1 - P)(x + da)$.)

It follows that $\Pi V = \text{Ker } p$ taken with the differential $-\Pi Pd$ is a homotopy fiber or a co-cone of the inclusion of complexes $i: K \to L = K \oplus V$.

Now, if we come back to our original setup where $i: K \to L$ is an inclusion of differential Lie superalgebras, we want to provide the cocomylinder $L \oplus \Pi V$ with a bracket extending the one in $L$ so that $j$ and $p$ will respect the brackets and $D$ be a derivation. It turns out that this

\textsuperscript{1} A homotopy-theoretic interpretation of our original construction with $\Delta$ was conjectured by an anonymous referee of this paper, who proposed to extend the brackets generated by $\Delta$ by formulae similar to (29)–(35) deduced below.
condition fixes the bracket in \( L \oplus IV \) uniquely. In addition to the original Lie bracket in \( L \), appear new brackets between elements of \( L \) and \( IV \), and inside \( IV \):

\[
[x, \Pi a] := (-1)^i \Pi P[x, a], \quad [\Pi a, \Pi b] := (-1)^{i+1} \Pi P[da, b].
\]

Up to the parity shift, the latter bracket is immediately recognizable as the beginning of our sequence of higher derived brackets generated by \( d \) in \( V \). The new binary bracket in \( L \oplus IV \) does not satisfy the Jacobi identity exactly; this gives rise to ternary brackets \( L \oplus IV \) of the form similar to the above, and so on. One can figure out the appearance of these higher brackets by an incomplete induction. Since in this paper we work with symmetric brackets, the final result is more conveniently formulated after a parity shift. Applying \( \Pi \) to (27) we get

\[
\xymatrix{ \Pi K \ar[r]^i & \Pi L \\ \Pi L \oplus V \ar[u]^j \ar[r]_p }
\]

which is a cocylinder diagram for \( i = i^{\Pi} : \Pi K \to \Pi L \) in the category of complexes. Here \( D = D^{\Pi} \) in \( \Pi L \oplus V \) is \( (\Pi x, a) \mapsto (-\Pi dx, P(x + da)) \). The desire to extend the bracket in \( \Pi L \) corresponding to the Lie bracket in \( L \) keeping \( D \) a derivation, naturally leads to the following definitions. The 0-ary bracket in \( \Pi L \oplus V \) is set to zero and as the unary bracket we take the operator \( D \):

\[
\{ \Pi x \} = -\Pi dx + Px, \tag{29}
\]

\[
\{ a \} = Pda. \tag{30}
\]

Then we define the binary brackets as

\[
\{ \Pi x, \Pi y \} = \Pi [x, y](-1)^i, \tag{31}
\]

\[
\{ \Pi x, a \} = P[x, a], \tag{32}
\]

\[
\{ a, b \} = P[da, b]. \tag{33}
\]

The higher-order brackets we define as

\[
\{ \Pi x, a_1, \ldots, a_n \} = P[\ldots [x, a_1], \ldots, a_n], \tag{34}
\]

\[
\{ a_1, \ldots, a_n \} = P[\ldots [da_1, a_2], \ldots, a_n]. \tag{35}
\]

where \( n > 1 \). All the other brackets except obtainable from these by symmetry, are set to zero. We arrive at a collection of odd symmetric multilinear operations on \( \Pi L \oplus V \). The subspace \( V \) is an ideal w.r.t. these operations and their restriction to \( V \) coincides with the higher derived brackets (24).

**Theorem 3.** Operations (29)–(35) make the space \( \Pi L \oplus V \) a strict \( L_\infty \)-algebra.
It follows from Theorem 3 that diagram (28) is a cocylinder diagram in the category of $L_\infty$-algebras, as it is clear that the maps $j$ and $p$ in (28) strictly respect the brackets, in particular giving $L_\infty$-maps. As a corollary we see that $V$ considered with the higher derived brackets (24) is a homotopy fiber of the inclusion of the (odd) differential Lie superalgebras $\Pi K \to \Pi L$. If we change the viewpoint at $L_\infty$-algebras and adopt a definition which differs from ours by the parity shift (see Remark 2.1), it will be possible to say that $\Pi V$ (with the corresponding ‘shifted’ higher derived brackets) is a homotopy fiber for the inclusion $K \to L$.

The proofs of Theorems 2 and 3, and other details, will be given elsewhere.\textsuperscript{2}

6. Discussion

A derived bracket (with this name) of two arguments appeared for the first time in the paper by Kosmann-Schwarzbach [20], who also referred to an unpublished text by Koszul of 1990. She proved that any odd derivation of a Loday (=Leibniz) algebra generates a new Loday bracket of the opposite parity by the formula $[a, b]_D = (-1)^{\tilde{a}}[Da, b]$. (The present author independently introduced a derived bracket around 1993 and proved a similar statement, in a slightly less generality than [20], namely, without the Loday algebras and working only with Lie superalgebras.) Unlike the brackets introduced in the present paper, the bracket $[a, b]_D$ does not necessarily satisfy (anti)symmetry even if the original bracket does. Antisymmetry is restored on suitable subspaces or quotient spaces provided the derived bracket can be restricted there. The present construction of higher derived brackets making use of a projector $P$ solves the problem by forcing the bracket to remain in a given subspace. The necessity to consider all the higher brackets, not just the binary bracket, is the price.

Retrospectively, binary derived brackets, considered on subspaces, can be recognized in many constructions of differential geometry, e.g., in the Cartan identities $[d, i_u] = \mathcal{L}_u, [\mathcal{L}_u, i_v] = i_{[u,v]}$ combined to give $i_{[u,v]} = [i_u, [d, i_v]]$. An important example is the coordinate-free expression for a Poisson bracket generated by a bivector field $B$ via the canonical Schouten bracket: $\{f, g\}_B = \{\{f, B\}, g\}$ up to a sign depending on conventions, and a similar expression for a Schouten structure via the canonical Poisson bracket. (For the author, these expressions were a starting point in the discovery of derived brackets.) Derived brackets have been also used for describing Lie algebroids (see, e.g., [35]) and Courant algebroids [30,28].

Higher brackets do not appear in these classical examples because for them the generating element is always, loosely speaking, ‘quadratic’: viz., a quadratic homological vector field $Q = \frac{1}{2} \sum_{i,j} Q^k_{ji} \partial_k$, a bivector field $B = \frac{1}{2} B^{ab}(x) x^b \wedge^a$, a quadratic Hamiltonian $S = \frac{1}{2} S^{ab}(x) p_b \wedge^a$, an odd Laplacian $\Delta$, etc. For the same reason there is no need to introduce a projector to remain in a chosen subspace of the ‘zero-order’ elements (such as vector fields with constant coefficients, functions on $M$ as opposed to those on $T^*M$, zero-order operators, etc.). On the other hand, a natural attempt to replace, say, a Poisson bivector field by an arbitrary multivector field satisfying $\{P, P\} = 0$ and still have a bracket on functions, requires introducing a projector and immediately leads to higher derived brackets and

\textsuperscript{2} See our new paper [36].
homotopy analogs of the classical examples. (See Example 4.2 and other examples in the previous section.)

The characterization of differential operators with the help of multiple commutators can be traced to Grothendieck [10]. Related to it the higher derived brackets of Example 4.4 essentially coincide with the operations $\Phi^r$ introduced by Koszul [22]. For a differential operator $\Delta$ on a graded commutative algebra, Koszul defined $\Phi^r_\Delta$ for $r > 0$ as

$$\Phi^r_\Delta(a_1 \otimes \cdots \otimes a_r) := m \circ (\Delta \otimes \text{id})\lambda^r(a_1 \otimes \cdots \otimes a_r),$$

where $\lambda^r(a_1 \otimes \cdots \otimes a_r) = (a_1 \otimes 1 - 1 \otimes a_1) \cdots (a_r \otimes 1 - 1 \otimes a_r)$ and $m$ stands for the multiplication; he stated that for each $r$, $\Phi^{r+1}_\Delta$ equals the failure of the Leibniz identity for $\Phi^r_\Delta$. He was basically interested in the binary operation $\Phi^2_\Delta$ generated by an odd operator of second order. It was stated in [22] that the failure of the homotopy Jacobi identity for $\Phi^2_\Delta$ (involving $\Phi^3_\Delta$) equaled $\Phi^3_\Delta$, and that the Leibniz identity for $\Delta$ and $\Phi^2_\Delta$ were equivalent to $\Phi^3_\Delta = 0$. Generalizations of Koszul’s operations $\Phi^r$ for various types of algebras, not necessarily commutative or associative were studied by Akman [1] (see also [2]). As in [22], she was mainly concerned with the binary bracket $\Phi^2$.

“Higher antibrackets” generated by an odd differential operator $\Delta$, together with higher Poisson brackets, have appeared in the series of physical papers [3–9] motivated by a development of the Batalin–Vilkovisky quantization. As Stasheff noted, they were also hiding in works on Batalin–Fradkin–Vilkovisky formalism, such as [19]. In [9] it was proved directly that the higher brackets defined by (22) form an $L_\infty$-algebra if $\Phi^2_\Delta = 0$. For “general antibrackets” on differential operators defined in [8] as the symmetrizations of multiple commutators

$$[[\ldots [[\Delta, A_1], A_2], \ldots, A_n]]$$

(no evaluation at 1, unlike (22)), where the operators $A_i$ are arbitrary and do not have to belong to an Abelian subalgebra, were obtained certain Jacobi-type identities more complicated than those for the $L_\infty$-algebras. Such algebraic structures are yet to be analyzed.

As we mentioned in Section 4, higher derived brackets of Example 4.4 make the natural framework for the problem of describing the generating operators of an odd bracket. Geometric constructions related with these ‘Batalin–Vilkovisky operators’ were considered in [11,32,21,13–15]. A complete picture was obtained in [16,17]. In [17], Khudaverdian and the author established a one-to-one correspondence between second-order differential operators on the algebra of densities $\mathfrak{B}(M)$ on a supermanifold $M$ and binary brackets in $\mathfrak{B}(M)$. For operators acting on functions, this specializes to a correspondence between operators and pairs consisting of a bracket on functions and an “upper connection” on volume forms [17]. For odd operators this gives a description of the Jacobi conditions in terms of this connection. Constructions of the present paper, hopefully, can be useful for generalizing the results of [17] to higher order operators.

There were suggested different approaches to Batalin–Vilkovisky algebras “up to homotopy”, as well as to homotopy Schouten (= Gerstenhaber) algebras. Operadic approaches to the latter are discussed in [34]. A direct definition of a homotopy Batalin–Vilkovisky algebra was suggested by Olga Kravchenko [23] and further generalized by Tamarkin and Tsygan [33]. In particular, besides the $L_\infty$-structure this definition provides
for the (strong) homotopy associativity of the product and homotopy Leibniz identities. The
examples in Section 4 satisfy much stricter conditions. On the other hand, an example of
higher brackets of differential operators as in [8] involves conditions that are weaker than
those of an \( L_\infty \)–structure. Therefore the final algebraic framework for these notions is yet
to be found.

The higher derived brackets that we introduced here are not the most general. A natural
extension of our constructions should be to allow the image of a projector \( P \) to be an arbitrary
Lie subalgebra, not necessarily Abelian. A condition generalizing (2) should then read

\[
P[a, b] = P[P a, b] + P[a, P b] - [P a, P b].
\]  

(Together with \( P[P a, P b] = [P a, P b] \) that means that both \( \text{Im } P \) and \( \ker P \) are subalge-
bras, i.e., the Lie superalgebra in question is the sum of two subalgebras.) In such case the
symmetry of higher derived brackets should remain only up to homotopy, and we should
end up with a yet more general notion of a (strongly) homotopy Lie algebra. In examples,
this should lead also to more general cases of homotopy Batalin–Vilkovisky algebras. For
instance, when \( P = 1 \), this should cover the “general antibrackets” of [8]. Projectors sat-
sifying (36) appeared in [12] with a totally different motivation. It was shown, remarkably,
that they come from operators on an associative algebra with a unit satisfying

\[
P 1 = 1, \quad P(a P b) = P(ab), \quad P((P a)b) = P a P b.
\]  

Khudaverdian’s result [12] is that upon conditions (37), the formal series

\[
\log(P e^a) = \log(1 + P a + \frac{1}{2} P(a^2) + \cdots)
\]

for any element \( a \) of the associative algebra can be expressed via commutators only and
the action of \( P \), thus obtaining a generalization of the Baker–Campbell–Hausdorff formula.
(These results were inspired by an analysis of certain Feynman diagrams in quantum field
theory. Examples, however, range to cobordism theory and Novikov’s operator doubles
[12].) There must be a connection with the present construction of higher derived brackets,
but it is yet to be understood. A question that is related, is to give an analogous “derived”
construction in the associative setting, leading to \( A_\infty \)-algebras and their relatives.

Another interesting direction of study should be derived brackets and homotopy algebras
arising from graded manifolds [35] (see also [29]).

We hope to consider these questions elsewhere.

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References


