AN INTRODUCTION TO THE ∞–LAPLACIAN

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Foreword

These lecture are an introduction to the analysis of $\infty-$harmonic functions, a subject that grew mature in recent years in the field of nonlinear partial differential equations. They correspond to the short courses I taught at the Universidade Federal do Ceará (Fortaleza, Brazil) in the (southern hemisphere) Summer of 2013 and at Aalto University (Helsinki, Finland) in the (northern hemisphere) Spring of 2013. A shorter version was used for a series of three lectures at KAUST (Thuwal, Saudi Arabia), early in 2017.

The material covered ranges from the Lipschitz extension problem to questions of existence, uniqueness and regularity for $\infty-$harmonic functions. A rigorous and detailed analysis of the equivalence between being absolutely minimising Lipschitz, enjoying comparison with cones and solving the ∞–Laplace equation in the viscosity sense is the backbone of the set of lectures. At the heart of the approach adopted lies the notion of comparison with cones, which is pivotal throughout the text. The proof of the existence of $\infty-$harmonic functions in the case of an unbounded domain, a few regularity results (including the Harnack inequality and the local Lipschitz continuity) and an easy proof, due to Armstrong and Smart, of the celebrated uniqueness theorem of Jensen complete the course.

My writing has been strongly influenced by the study of the survey papers of Crandall [3] and Aronsson–Crandall–Juutinen [2] and I claim no originality whatsoever. Having evolved from my handwritten notes upon studying those sources, it is only natural that some of the material is reproduced almost verbatim, including some of the problems proposed as exercises. My sole contributions are in the level of detail of some of the proofs (*the devil is frequently there*), the simplification of a few arguments and the organisation of the text. I believe the choice of the topics covered is adequate, for example, for the final part of an introductory graduate course on nonlinear pdes.

I benefited enormously from the interaction with the excellent students that attended the course in Fortaleza, Helsinki and Thuwal, and the interesting discussions on the topic with many colleagues. I mention, in particular, Diogo Gomes, Eduardo Teixeira, Juha Kinnunen, Juha Videman, Juhana Siljander, Levon Nurbekyan, Mikko Parviainen and Tuomo Kuusi, whose input influenced directly the writing of some of the proofs. I also had the chance to exchange opinions with Petri Juutinen, who introduced me to the subject back in 1999 when, as postdoc students, we shared an office at Northwestern University, in Chicago. Any typos or inaccuracies that remain are, of course, my full responsibility.

I warmly thank Eduardo Teixeira and Fábio Montenegro, at UFC, Juha Kinnunen and Tuomo Kuusi, at Aalto, and Diogo Gomes, at KAUST, for the opportunity to teach the course and the wonderful hospitality in Fortaleza, Helsinki and Thuwal.

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Contents

1. The Lipschitz extension problem

We start from the very beginning, with the basic definition of Lipschitz function.

Definition 1. *Let $X \subset \mathbb{R}^n$. A function $f : X \to \mathbb{R}$ is Lipschitz continuous on X, equivalently $f \in Lip(X)$, if there exists a constant $L \in [0, \infty)$ such that*

$$|f(x) - f(y)| \leq L\,|x - y|, \quad \forall x, y \in X. \tag{1}$$

Any $L \in [0, \infty)$ for which (1) holds is called a Lipschitz constant for f in X. The least constant $L \in [0, \infty)$ for which (1) holds is denoted by $\mathrm{Lip}_f(X)$.

If there is no L for which (1) holds, we write $\mathrm{Lip}_f(X) = \infty$.

Let $U \subset \mathbb{R}^n$ be open and bounded and denote its boundary with ∂U. We will be concerned with the problem of extending a Lipschitz function defined on ∂U to $\overline{U}$ without increasing its Lipschitz constant. Since decreasing it is out of the question, the best we can hope for is to keep it the same.

The Lipschitz Extension Problem. Given $f \in Lip(\partial U)$, find $u \in Lip(\overline{U})$ such that

$$u = f \ \text{ on } \partial U \qquad \text{and} \qquad \mathrm{Lip}_u(\overline{U}) = \mathrm{Lip}_f(\partial U).$$

In fact, we are both extending the function and minimising the Lipschitz constant.

If $y, z \in \partial U$ and $x \in \overline{U}$, then any Lipschitz extension u of f trivially satisfies

$$f(z) - \mathrm{Lip}_f(\partial U)|x - z| \leq u(x) \leq f(y) + \mathrm{Lip}_f(\partial U)|x - y|$$

since $f(z) = u(z)$ and $f(y) = u(y)$. Let us show that these two bounds belong to $Lip(\overline{U})$.

Let $z \in \partial U$ and put

$$F_z(x) = f(z) - \mathrm{Lip}_f(\partial U)|x - z|, \quad x \in \overline{U}.$$

We then have, for any $x, \tilde{x} \in \overline{U}$,

$$\begin{aligned}
|F_z(x) - F_z(\tilde{x})| &= \big|f(z) - \mathrm{Lip}_f(\partial U)|x - z| - f(z) + \mathrm{Lip}_f(\partial U)|\tilde{x} - z|\big| \\
&= \mathrm{Lip}_f(\partial U)\ ||\tilde{x} - z| - |x - z|| \\
&\leq \mathrm{Lip}_f(\partial U)\ |\tilde{x} - z - x + z| \\
&= \mathrm{Lip}_f(\partial U)\ |x - \tilde{x}|\,.
\end{aligned}$$

This means that $F_z \in Lip(\overline{U})$ and that $\mathrm{Lip}_f(\partial U)$ is a Lipschitz constant for F_z in $\overline{U}$. Since $\mathrm{Lip}_f(\partial U)$ is independent of z it is a common Lipschitz constant for all F_z, $z \in \partial U$.

Given $y \in \partial U$, an entirely analogous reasoning holds for

$$G_y(x) = f(y) + \mathrm{Lip}_f(\partial U)|x - y|, \quad x \in \overline{U}.$$

Definition 2. *The MacShane-Whitney extensions of $f \in Lip(\partial U)$ are the functions defined in $\overline{U}$ by*

$$\mathcal{MW}_*(f)(x) := \sup_{z \in \partial U} F_z(x) = \sup_{z \in \partial U} \left\{ f(z) - \mathrm{Lip}_f(\partial U)|x - z| \right\}$$

and

$$\mathcal{MW}^*(f)(x) := \inf_{y \in \partial U} G_y(x) = \inf_{y \in \partial U} \left\{ f(y) + \mathrm{Lip}_f(\partial U)|x - y| \right\}.$$

Since both the infimum and the supremum of a family of Lipschitz functions, with a fixed Lipschitz constant, is still Lipschitz and has, if it is finite, the same Lipschitz constant, we conclude that both $\mathcal{MW}_*(f)$ and $\mathcal{MW}^*(f)$ are Lipschitz functions in $\overline{U}$, with Lipschitz constant $\mathrm{Lip}_f(\partial U)$.

We next show that $\mathcal{MW}_*(f) = f$ on ∂U (the same holds, of course, for $\mathcal{MW}^*(f)$). Let $x \in \partial U$. Then

$$\mathcal{MW}_*(f)(x) \geq F_x(x) = f(x) - \mathrm{Lip}_f(\partial U)|x - x| = f(x).$$

On the other hand, since $f \in Lip(\partial U)$,

$$f(z) - \mathrm{Lip}_f(\partial U)|x - z| \leq f(x),$$

for any $z \in \partial U$, and thus

$$\mathcal{MW}_*(f)(x) = \sup_{z \in \partial U} \left\{ f(z) - \mathrm{Lip}_f(\partial U)|x - z| \right\} \leq f(x).$$

This implies that

$$\mathrm{Lip}_{\mathcal{MW}_*(f)}(\overline{U}) = \mathrm{Lip}_{\mathcal{MW}^*(f)}(\overline{U}) = \mathrm{Lip}_f(\partial U).$$

We have just proved the following result.

Theorem 1. *The MacShane-Whitney extensions, $\mathcal{MW}_*(f)$ and $\mathcal{MW}^*(f)$, solve the Lipschitz extension problem for $f \in Lip(\partial U)$ and if u is any other solution to the problem then*

$$\mathcal{MW}_*(f) \leq u \leq \mathcal{MW}^*(f) \quad \text{in } \overline{U}.$$

The Lipschitz Extension Problem is then uniquely solvable if

$$\mathcal{MW}_*(f) = \mathcal{MW}^*(f) \quad \text{in } \overline{U},$$

which rarely happens.

Example 1. *Let $n = 1$ and $U = (-1,0) \cup (0,1)$. Consider $f : \partial U \to \mathbb{R}$ defined by $f(-1) = f(0) = 0$ and $f(1) = 1$. Then* $\mathrm{Lip}_f(\partial U) = 1$ *and a simple computation gives*

$$\mathcal{MW}_*(f)(x) = \begin{cases} -x-1 & \text{if } -1 \le x \le -\frac{1}{2} \\ \\ x & \text{if } -\frac{1}{2} \le x \le 1 \end{cases}$$

and

$$\mathcal{MW}^*(f)(x) = \begin{cases} x+1 & \text{if } -1 \le x \le -\frac{1}{2} \\ \\ |x| & \text{if } -\frac{1}{2} \le x \le 1, \end{cases}$$

which are, of course, different functions.

This lack of uniqueness in the Lipschitz extension problem being an issue, other features are perhaps even more relevant. We will illustrate them with the help of the above example.

Non comparison: Take as boundary data $g : \partial U \to \mathbb{R}$ defined by $g(-1) = 0$, $g(0) = \frac{1}{2}$ and $g(1) = 1$. Then $\mathrm{Lip}_g(\partial U) = \frac{1}{2}$ and we easily see that $\mathcal{MW}_*(g) = \mathcal{MW}^*(g)$, so the problem is uniquely solvable. But we have $f \le g$ and, nevertheless, neither

$$\mathcal{MW}^*(f) \le \mathcal{MW}^*(g)$$

nor

$$\mathcal{MW}^*(f) \ge \mathcal{MW}^*(g)$$

hold.

Non stability: Let $V = \left(-\frac{3}{4}, -\frac{1}{4}\right)$. Then $\mathcal{MW}^*(f)|_{\partial V} \equiv \frac{1}{4}$ and so also

$$\mathcal{MW}^*\left(\mathcal{MW}^*(f)|_{\partial V}\right) \equiv \frac{1}{4} \neq \mathcal{MW}^*(f) \quad \text{in } V.$$

In particular, a repeated application of the MacShane-Whitney extension decreases the local Lipschitz constant.

Non locality: Again let $V = \left(-\frac{3}{4}, -\frac{1}{4}\right)$; then

$$\mathrm{Lip}_{\mathcal{MW}^*(f)}(V) = 1 \neq 0 = \mathrm{Lip}_{\mathcal{MW}^*(f)}(\partial V).$$

The extension defined by

$$u(x) = \begin{cases} 0 & \text{if} \quad -1 \leq x \leq 0 \\ x & \text{if} \quad 0 \leq x \leq 1 \end{cases}$$

satisfies this property for any $V \subset\subset U$ (this means $\overline{V}$ is a compact subset of U). In a certain sense it locally varies as little as possible.

The notion of locality is embedded in the following definition that meets the need to define a sort of *canonical* Lipschitz extension, which we will eventually prove is unique.

Definition 3. *A function $u \in C(U)$ is absolutely minimising Lipschitz on U, and we write $u \in \mathrm{AML}(U)$, if*

$$\mathrm{Lip}_u(V) = \mathrm{Lip}_u(\partial V), \quad \forall\, V \subset\subset U. \tag{2}$$

This notion is trivially local in the sense that if $u \in \mathrm{AML}(U)$ and $V \subset U$ then $u \in \mathrm{AML}(V)$. It does not involve boundary conditions, it is a property of continuous functions defined on open sets alone.

Still we can try to recast the Lipschitz extension problem as the following problem: given $f \in Lip(\partial U)$, find $u \in C(\overline{U})$ such that

$$u \in \mathrm{AML}(U) \qquad \text{and} \qquad u = f \ \text{ on } \partial U. \tag{3}$$

It can be shown that a solution to this problem satisfies the Lipschitz extension problem.

2. Comparison with cones

We now introduce a more geometric notion, that of comparison with cones. It will be instrumental in most of the analysis hereafter.

Definition 4. *A cone with vertex $x_0 \in \mathbb{R}^n$ is a function of the form*

$$C(x) = a + b|x - x_0|, \quad a, b \in \mathbb{R}.$$

The height of C is a and its slope is b.

Definition 5. *For a cone C with vertex at x_0, the half-line*

$$\{x_0 + t(x - x_0),\ t \geq 0\}$$

is the ray of C through the point x.

Lemma 1. *If a set V contains two distinct points on the same ray of a cone C with slope b, then*

$$\mathrm{Lip}_C(V) = |b|.$$

Proof. Let $C(x) = a + b|x - x_0|$. Then, for any $x, y \in \mathbb{R}^n$,

$$\frac{|C(x) - C(y)|}{|x - y|} = |b| \frac{\Big||x - x_0| - |y - x_0|\Big|}{|x - y|} \leq |b|,$$

so $|b|$ is a Lipschitz constant for C in any set.

If w, y are distinct points on the same ray of C, we have, for a certain x^*, $y = x_0 + \alpha(x^* - x_0)$ and $w = x_0 + \beta(x^* - x_0)$, with $\alpha, \beta \geq 0$, $\alpha \neq \beta$. Then

$$\begin{aligned}\frac{|C(y) - C(w)|}{|y - w|} &= \frac{|C(x_0 + \alpha(x^* - x_0)) - C(x_0 + \beta(x^* - x_0))|}{|x_0 + \alpha(x^* - x_0) - x_0 - \beta(x^* - x_0)|} \\ &= \frac{\Big|a + b|x_0 + \alpha(x^* - x_0) - x_0| - a - b|x_0 + \beta(x^* - x_0) - x_0|\Big|}{|\alpha - \beta||x^* - x_0|} \\ &= |b| \frac{|\alpha - \beta||x^* - x_0|}{|\alpha - \beta||x^* - x_0|} = |b|,\end{aligned}$$

and if $w, y \in V$ then $\mathrm{Lip}_C(V) = |b|$. □

Corollary 1. *Let $V \subset \mathbb{R}^n$ be non-empty and open, and C be a cone with slope b. Then*

$$\mathrm{Lip}_C(V) = |b|.$$

Moreover, if V is bounded and does not contain the vertex of C, then

$$\mathrm{Lip}_C(\partial V) = |b|.$$

Definition 6. *A function $w \in C(U)$ enjoys comparison with cones from above in U if, for every $V \subset\subset U$ and every cone C whose vertex is not in V,*

$$w \leq C \text{ on } \partial V \quad \Longrightarrow \quad w \leq C \text{ in } V.$$

A function w enjoys comparison with cones from below if $-w$ enjoys comparison with cones from above. A function w enjoys comparison with cones if it enjoys comparison with cones from above and from below.

Lemma 2. *The following is an equivalent condition to $u \in C(U)$ enjoying comparison with cones from above in U: for every $V \subset\subset U$, $b \in \mathbb{R}$ and $z \notin V$,*

$$u(x) - b|x-z| \leq \max_{w \in \partial V} \left(u(w) - b|w-z| \right), \quad \forall\, x \in V.$$

Proof. To prove the necessity of the condition, let $V \subset\subset U$, $b \in \mathbb{R}$ and $z \notin V$. We trivially have

$$u(x) - b|x-z| \leq \max_{w \in \partial V} \left(u(w) - b|w-z| \right), \quad \forall\, x \in \partial V. \tag{4}$$

This can be rewritten as

$$u(x) \leq C(x) := \max_{w \in \partial V} \left(u(w) - b|w-z| \right) + b|x-z|, \quad \forall\, x \in \partial V,$$

for the cone C centred at $z \notin V$. Since u enjoys comparison with cones from above in U, (4) also holds for any $x \in V$.

Reciprocally, let $V \subset\subset U$ and let

$$C(x) = a + b|x-z|, \quad \text{with } a, b \in \mathbb{R},$$

be any cone with vertex at $z \notin V$ such that $u \leq C$ on ∂V. We know that, for every $x \in V$,

$$\begin{aligned} u(x) - b|x-z| &\leq \max_{w \in \partial V} \left(u(w) - b|w-z| \right) \\ \Rightarrow u(x) - a - b|x-z| &\leq \max_{w \in \partial V} \left(u(w) - a - b|w-z| \right) \\ \Rightarrow u(x) - C(x) &\leq \max_{w \in \partial V} \left(u(w) - C(w) \right) \leq 0 \end{aligned}$$

since $u \leq C$ on ∂V. We conclude that also $u \leq C$ in V. □

3. Comparison with cones and absolutely minimising Lipschitz

One of the main results in these notes is the following equivalence between being absolutely minimising Lipschitz and enjoying comparison with cones.

Theorem 2. *A function $u \in C(U)$ is absolutely minimising Lipschitz in U if, and only if, it enjoys comparison with cones in U.*

Proof. We start with the sufficiency. Suppose u enjoys comparison with cones in U and let $V \subset\subset U$. We want to show that

$$\mathrm{Lip}_u(V) = \mathrm{Lip}_u(\partial V).$$

Since $u \in C(\overline{V})$, we have $\mathrm{Lip}_u(V) = \mathrm{Lip}_u(\overline{V})$ (see Problem 1). Then, as $\partial V \subset \overline{V}$, we trivially have that $\mathrm{Lip}_u(V) \geq \mathrm{Lip}_u(\partial V)$ and it remains to prove the other inequality.

First, observe that, for any $x \in V$,

$$\mathrm{Lip}_u\left(\partial\left(V \setminus \{x\}\right)\right) = \mathrm{Lip}_u(\partial V \cup \{x\}) = \mathrm{Lip}_u(\partial V). \tag{5}$$

To see this holds we need only check that, for any $y \in \partial V$,

$$|u(y) - u(x)| \leq \mathrm{Lip}_u(\partial V)\,|y - x|,$$

which is equivalent to

$$u(y) - \mathrm{Lip}_u(\partial V)\,|x - y| \leq u(x) \leq u(y) + \mathrm{Lip}_u(\partial V)\,|x - y|. \tag{6}$$

This clearly holds for any $x \in \partial V$ but what we want to prove is that it holds for $x \in V$. Let's focus on the second inequality in (6). The right-hand side can be regarded as the cone

$$C(x) = u(y) + \mathrm{Lip}_u(\partial V)\,|x - y|,$$

centred at $y \in \partial V$. Since $y \notin V$ and u enjoys comparison with cones from above in U, the inequality holds in V because it holds on ∂V. To obtain the first inequality in (6), we argue analogously, using comparison with cones from below.

Now let $x, y \in V$. Using (5) twice, we obtain

$$\mathrm{Lip}_u(\partial V) = \mathrm{Lip}_u\left(\partial\left(V \setminus \{x\}\right)\right) = \mathrm{Lip}_u\left(\partial\left(V \setminus \{x, y\}\right)\right).$$

Since $x, y \in \partial\left(V \setminus \{x, y\}\right) = \partial V \cup \{x, y\}$, we have

$$|u(x) - u(y)| \leq \mathrm{Lip}_u\left(\partial\left(V \setminus \{x, y\}\right)\right)\,|x - y| = \mathrm{Lip}_u(\partial V)\,|x - y|.$$

Thus

$$\mathrm{Lip}_u(V) \leq \mathrm{Lip}_u(\partial V).$$

Now the necessity. Suppose $u \in \mathrm{AML}(U)$. For $V \subset\subset U$, we have

$$\mathrm{Lip}_u(V) = \mathrm{Lip}_u(\partial V).$$

Due to Lemma 2, we want to prove that for every $b \in \mathbb{R}$ and $z \notin V$,

$$u(x) - b|x - z| \leq \max_{w \in \partial V}\left(u(w) - b|w - z|\right), \quad \forall\, x \in V.$$

Setting

$$W = \left\{x \in V \ : \ u(x) - b|x - z| > \max_{w \in \partial V}\left(u(w) - b|w - z|\right)\right\},$$

the result will follow by proving that $W = \emptyset$. We will argue by contradiction. Consider the cone

$$C(x) := \max_{w \in \partial V}\left(u(w) - b|w - z|\right) + b|x - z|.$$

Then $W = V \cap (u-C)^{-1}((0,\infty))$ is open and

$$u = C \quad \text{on} \quad \partial W. \tag{7}$$

To prove this, note first that, trivially, if $x \in \partial V$ then $(u-C)(x) \leq 0$. Now suppose $x \in \partial W$, with $(u-C)(x) > 0$. Then $x \notin \partial V$, and since $\partial W \subset \overline{V}$, $x \in V$, in which case $x \in W$, which is a contradiction since W is open. If $x \in \partial W$, with $(u-C)(x) < 0$, then, since $u - C \in C(U)$, there is a neighbourhood N_x of x such that $u - C < 0$ in N_x. So $N_x \cap W = \emptyset$, again a contradiction.

We then have, since $u \in \mathrm{AML}(U)$,

$$\mathrm{Lip}_u(W) = \mathrm{Lip}_u(\partial W) = \mathrm{Lip}_C(\partial W) = |b|,$$

due to Corollary 1, because $z \notin W$, since $z \notin V$ and $W \subset V$.

Take $x_0 \in W$. The ray of C through x_0

$$\{z + t(x_0 - z),\ t \geq 0\}$$

contains a segment in W, containing x_0, which meets ∂W at its endpoints. Consider the functions

$$F(t) = C(z + t(x_0 - z)) = a + b|x_0 - z|\, t, \quad t \geq 0,$$

with $a = \max_{w \in \partial V}(u(w) - b|w - z|)$, and

$$G(t) = u(z + t(x_0 - z)), \quad t \geq 0.$$

They coincide at the endpoints of the segment, since $u = C$ on ∂W. Now F is affine with slope $|b||x_0 - z|$, while G has $|b||x_0 - z|$ as Lipschitz constant on the segment. In fact,

$$\begin{aligned} \frac{|G(t_1) - G(t_2)|}{|t_1 - t_2|} &= \frac{|u(z + t_1(x_0 - z)) - u(z + t_2(x_0 - z))|}{|t_1 - t_2|} \\ &\leq \frac{|b||(t_1 - t_2)(x_0 - z)|}{|t_1 - t_2|} = |b||x_0 - z|, \end{aligned}$$

because $\mathrm{Lip}_u(W) = |b|$ and the segment is contained in W. We conclude that F and G are the same function on the segment and, since it contains x_0,

$$G(1) = u(x_0) = C(x_0) = F(1).$$

We have reached a contradiction because $x_0 \in W$ and so $u(x_0) > C(x_0)$.

The proof that u satisfies comparison with cones from below in U is analogous and uses a lemma similar to Lemma 2. □

4. The $\infty-$Laplacian

We now turn to the connection with $\infty-$harmonic functions.

Definition 7. *The partial differential operator given, on smooth functions φ, by*

$$\Delta_\infty \varphi := \sum_{i,j=1}^{n} \varphi_{x_i}\varphi_{x_j}\varphi_{x_i x_j} = \langle D^2\varphi D\varphi, D\varphi\rangle$$

is called the $\infty-$Laplacian.

This operator is not in divergence form so we can not (formally) integrate by parts to define a notion of weak solution. The appropriate notion to consider is that of viscosity solution.

Definition 8. *A function $w \in C(U)$ is a viscosity subsolution of $\Delta_\infty u = 0$ (or a viscosity solution of $\Delta_\infty u \geq 0$ or $\infty-$subharmonic) in U if, for every $\hat{x} \in U$ and every $\varphi \in C^2(U)$ such that $w - \varphi$ has a local maximum at $\hat{x}$, we have*

$$\Delta_\infty \varphi(\hat{x}) \geq 0.$$

A function $w \in C(U)$ is $\infty-$superharmonic in U if $-w$ is $\infty-$subharmonic in U. A function $w \in C(U)$ is $\infty-$harmonic in U if it is both $\infty-$subharmonic and $\infty-$superharmonic in U.

Lemma 3. *If $u \in C^2(U)$ then u is $\infty-$harmonic in U if, and only if, $\Delta_\infty u = 0$ in the pointwise sense.*

Proof. Suppose u is $\infty-$harmonic. Then it is $\infty-$subharmonic and we take $\varphi = u$ in the definition. Since every point $x \in U$ will then be a local maximum of $u - \varphi \equiv 0$, $\Delta_\infty u(x) \geq 0$, for every $x \in U$. Since also $-u$ is $\infty-$subharmonic, we get in addition

$$\Delta_\infty(-u)(x) \geq 0 \;\Leftrightarrow\; -\Delta_\infty u(x) \geq 0 \;\Leftrightarrow\; \Delta_\infty u(x) \leq 0, \quad \forall x \in U$$

and so $\Delta_\infty u = 0$ in the pointwise sense.

Reciprocally, suppose $\Delta_\infty u = 0$ in the pointwise sense and take $\hat{x} \in U$ and $\varphi \in C^2(U)$ such that $u - \varphi$ has a local maximum at $\hat{x}$. We want to prove that $\Delta_\infty \varphi(\hat{x}) \geq 0$, thus showing that u is $\infty-$subharmonic (the $\infty-$superharmonicity is obtained in an analogous way). We have, since $u - \varphi \in C^2(U)$ and $\hat{x} \in U$ is a local maximum,

$$D(u - \varphi)(\hat{x}) = 0 \;\Leftrightarrow\; Du(\hat{x}) = D\varphi(\hat{x})$$

and

$$D^2(u-\varphi)(\hat{x}) \preceq 0 \;\Leftrightarrow\; \langle D^2u(\hat{x})\xi,\xi\rangle \le \langle D^2\varphi(\hat{x})\xi,\xi\rangle, \quad \forall x \in \mathbb{R}^n.$$

Then

$$\begin{aligned} \Delta_\infty\varphi(\hat{x}) &= \langle D^2\varphi(\hat{x})D\varphi(\hat{x}), D\varphi(\hat{x})\rangle \\ &\ge \langle D^2u(\hat{x})D\varphi(\hat{x}), D\varphi(\hat{x})\rangle \\ &= \langle D^2u(\hat{x})Du(\hat{x}), Du(\hat{x})\rangle \\ &= \Delta_\infty u(\hat{x}) \\ &= 0. \end{aligned}$$

□

We now show that the celebrated flatland example of Aronsson

$$u(x,y) = x^{\frac{4}{3}} - y^{\frac{4}{3}}$$

is $\infty-$subharmonic in $\mathbb{R}^2$. The proof that it is also $\infty-$superharmonic is analogous.

Take any point $(x_0,y_0) \in \mathbb{R}^2$ and $\varphi \in C^2(\mathbb{R}^2)$ such that $u-\varphi$ has a local maximum at (x_0,y_0). We start by observing that, since $u \in C^1(\mathbb{R}^2)$,

$$D(u-\varphi)(x_0,y_0) = 0$$

and, consequently,

$$\varphi_x(x_0,y_0) = u_x(x_0,y_0) = \frac{4}{3}x_0^{\frac{1}{3}} \tag{8}$$

and

$$\varphi_y(x_0,y_0) = u_y(x_0,y_0) = -\frac{4}{3}y_0^{\frac{1}{3}}. \tag{9}$$

We first exclude the case $x_0 = 0$. If $\varphi \in C^2(\mathbb{R}^2)$ is such that $u-\varphi$ has a local maximum at $(0,y_0)$, then

$$\begin{aligned} & (u-\varphi)(x,y_0) \le (u-\varphi)(0,y_0) \\ \Leftrightarrow\quad & x^{\frac{4}{3}} \le \varphi(x,y_0) - \varphi(0,y_0), \end{aligned} \tag{10}$$

for every x in a neighbourhood of 0 and this simply can not happen. In fact, letting $F(x) = \varphi(x,y_0) - \varphi(0,y_0)$, we have $F(0) = 0$ and also

$$F'(0) = \varphi_x(0,y_0) = u_x(0,y_0) = 0.$$

Then, by Taylor's theorem,

$$\lim_{x\to 0}\frac{F(x)}{x^2} = \frac{F''(0)}{2} = \frac{\varphi_{xx}(0,y_0)}{2} < +\infty.$$

On the other hand, if (10) would hold,

$$\lim_{x\to 0}\frac{F(x)}{x^2}\geq \lim_{x\to 0}\frac{x^{\frac{4}{3}}}{x^2}=\lim_{x\to 0}x^{-\frac{2}{3}}=+\infty,$$

a contradiction.

We next consider the case $x_0\neq 0$ and $y_0=0$. If $\varphi\in C^2(\mathbb{R}^2)$ is such that $u-\varphi$ has a local maximum at $(x_0,0)$, then

$$\begin{aligned}&(u-\varphi)(x,0)\leq (u-\varphi)(x_0,0)\\ \Leftrightarrow\quad & x^{\frac{4}{3}}-\varphi(x,0)\leq x_0^{\frac{4}{3}}-\varphi(x_0,0),\end{aligned}\tag{11}$$

for every x in a neighbourhood of x_0. This means that the function

$$G(x)=x^{\frac{4}{3}}-\varphi(x,0)$$

has a local maximum at the point x_0. Since it is of class C^2 in a neighbourhood of x_0 (small enough that it does not contain 0), we have $G'(x_0)=0$ and

$$G''(x_0)\leq 0\quad\Leftrightarrow\quad \varphi_{xx}(x_0,0)\geq \frac{4}{9}x_0^{-\frac{2}{3}}\geq 0.\tag{12}$$

Then, using (8), (9) and (12),

$$\begin{aligned}\Delta_\infty\varphi(x_0,0)&=\Big(\varphi_x^2\varphi_{xx}+2\varphi_x\varphi_y\varphi_{xy}+\varphi_y^2\varphi_{yy}\Big)(x_0,0)\\&=\varphi_x^2(x_0,0)\varphi_{xx}(x_0,0)\geq 0\end{aligned}$$

as required.

Finally, if both $x_0\neq 0$ and $y_0\neq 0$, u is C^2 in a neighbourhood of (x_0,y_0) and the equation is satisfied in the pointwise sense, the calculation being trivial.

5. Comparison with cones and $\infty-$harmonic

A crucial fact for $\infty-$harmonic functions is that they can be characterised through comparison with cones.

Theorem 3. *If $u\in C(U)$ is $\infty-$subharmonic then it enjoys comparison with cones from above.*

Proof. According to Lemma 2, we want to prove that, given $V\subset\subset U$, $b\in\mathbb{R}$ and $z\notin V$,

$$u(x)-b|x-z|\leq \max_{w\in\partial V}\left(u(w)-b|w-z|\right),\quad \forall\, x\in V.\tag{13}$$

Note that if G is smooth, we have

$$\Delta_\infty G(|x|)=G''(|x|)\left[G'(|x|)\right]^2,\quad x\neq 0.$$

Taking $G(t)=bt-\gamma t^2$, we have, for all $x\in V$ (recall that $z\notin V$),

$$\begin{aligned}\Delta_\infty\left(b|x-z|-\gamma|x-z|^2\right) &= \Delta_\infty G(|x-z|)\\ &= G''(|x-z|)\left[G'(|x-z|)\right]^2\\ &= -2\gamma\left(b-2\gamma|x-z|\right)^2\\ &< 0\end{aligned}$$

if $\gamma>0$ is small enough. In particular, since V is bounded, we must have, if $b>0$ (the case $b\leq 0$ is trivial),

$$\gamma<\frac{b}{2\sup\limits_{x\in V}|x-z|}.$$

Now, since u is $\infty-$subharmonic in $V\subset\subset U$ (due to the local character of the notion of viscosity subsolution),

$$u(x)-\left(b|x-z|-\gamma|x-z|^2\right)$$

can not have a local maximum in V. Then

$$u(x)-\left(b|x-z|-\gamma|x-z|^2\right)\leq\max_{w\in\partial V}\left(u(w)-\left(b|w-z|-\gamma|w-z|^2\right)\right),$$

for all $x\in V$. Finally, let $\gamma\to 0$ to obtain (13) and thus the result. $\square$

Magnificently, the reciprocal also holds.

Theorem 4. *If $u\in C(U)$ enjoys comparison with cones from above then it is $\infty-$subharmonic.*

Proof. We start by observing that, for every $x\in B_r(y)\subset\subset U$,

$$u(x)\leq u(y)+\max_{w\in\partial B_r(y)}\left(\frac{u(w)-u(y)}{r}\right)|x-y|. \tag{14}$$

The inequality clearly holds for $x\in\partial\left(B_r(y)\setminus\{y\}\right)=\partial B_r(y)\cup\{y\}$ and, since the right-hand side is a cone with vertex at $y\notin B_r(y)\setminus\{y\}$, the open set $B_r(y)\setminus\{y\}\subset\subset U$ and u enjoys comparison with cones from above, it also holds for $x\in B_r(y)\setminus\{y\}$; that it holds for $x=y$ is trivial.

Now, we rewrite (14) as

$$u(x)-u(y)\leq\max_{w\in\partial B_r(y)}\left(u(w)-u(x)\right)\frac{|x-y|}{r-|x-y|}. \tag{15}$$

This is just algebra:

$$\begin{aligned}&u(x)\leq u(y)+\max_{w\in\partial B_r(y)}\left(\frac{u(w)-u(y)}{r}\right)|x-y|\\ \Leftrightarrow\ &u(x)\leq u(y)+\left(\max_{w\in\partial B_r(y)}u(w)-u(y)\right)\frac{|x-y|}{r}\end{aligned}$$

$$\Leftrightarrow\; u(x) - \frac{r-|x-y|}{r}u(y) \le \max_{w\in\partial B_r(y)} u(w)\frac{|x-y|}{r}$$
$$\Leftrightarrow\; \frac{r}{r-|x-y|}u(x) - u(y) \le \max_{w\in\partial B_r(y)} u(w)\frac{|x-y|}{r-|x-y|}$$
$$\Leftrightarrow\; \left(1+\frac{|x-y|}{r-|x-y|}\right)u(x) - u(y) \le \max_{w\in\partial B_r(y)} u(w)\frac{|x-y|}{r-|x-y|}$$
$$\Leftrightarrow\; u(x) - u(y) \le \max_{w\in\partial B_r(y)} \left(u(w)-u(x)\right)\frac{|x-y|}{r-|x-y|}.$$

We first prove the result at points of twice differentiability. If u is twice continuously differentiable at x_0, namely if there is a $p\in\mathbb{R}^n$ and a symmetric $n\times n$ matrix X such that

$$u(z) = u(x_0) + \langle p, z-x_0\rangle + \frac{1}{2}\langle X(z-x_0), z-x_0\rangle + o(|z-x_0|^2), \tag{16}$$

so that

$$p = Du(x_0) \qquad \text{and} \qquad X = D^2u(x_0),$$

we show that

$$\Delta_\infty u(x_0) = \langle Xp, p\rangle \ge 0.$$

Let $x_0\in U$ be a point of twice differentiability for u. Choose

$$r < \frac{1}{2}\mathrm{dist}(x_0, \partial U)$$

and λ small enough so that, for $y_0 = x_0 - \lambda Du(x_0)$, $B_r(y_0)\subset\subset U$ and

$$x_0\in B_r(y_0) \;\Leftrightarrow\; |x_0-y_0|\le r \;\Leftrightarrow\; \lambda \le \frac{r}{|Du(x_0)|}.$$

Put $z=y_0$, in (16) to obtain, with $p = Du(x_0)$,

$$u(y_0) = u(x_0) + \langle p, -\lambda p\rangle + \frac{1}{2}\langle X(-\lambda p), -\lambda p\rangle + o(|-\lambda p|^2)$$
$$\Leftrightarrow\; u(x_0) - u(y_0) = \lambda|p|^2 - \frac{1}{2}\lambda^2\langle Xp, p\rangle - o(\lambda^2|p|^2).$$

Then, let $w_{r,\lambda}\in\partial B_r(y_0)$ be such that

$$u(w_{r,\lambda}) = \max_{w\in\partial B_r(y_0)} u(w)$$

and put $z = w_{r,\lambda}$ in (16) to obtain

$$u(w_{r,\lambda}) - u(x_0) = \langle p, w_{r,\lambda}-x_0\rangle + \frac{1}{2}\langle X(w_{r,\lambda}-x_0), w_{r,\lambda}-x_0\rangle + o(|w_{r,\lambda}-x_0|^2).$$

Now, choose $x=x_0$ and $y=y_0$ in (15) to get, after division by λ,

$$|p|^2 - \frac{1}{2}\lambda\langle Xp,p\rangle - o(\lambda)$$
$$\le \left(\langle p, w_{r,\lambda}-x_0\rangle + \frac{1}{2}\langle X(w_{r,\lambda}-x_0), w_{r,\lambda}-x_0\rangle + o((r+\lambda|p|)^2)\right)\frac{|p|}{r-\lambda|p|}.$$

Note that

$$|w_{r,\lambda} - x_0| = |w_{r,\lambda} - y_0 - \lambda p| \leq r + \lambda|p|.$$

We now send $\lambda \downarrow 0$ to get

$$\begin{aligned} |p|^2 &\leq \left(\left\langle p, \frac{w_r - x_0}{r} \right\rangle + \frac{1}{2} \left\langle X \left(\frac{w_r - x_0}{r} \right), w_r - x_0 \right\rangle \right) |p| + |p| o(r) \\ &\leq |p|^2 + \frac{1}{2} \left\langle X \left(\frac{w_r - x_0}{r} \right), w_r - x_0 \right\rangle |p| + |p| o(r), \end{aligned} \tag{17}$$

where $w_r \in \partial B_r(x_0)$ is any limit point of $w_{r,\lambda}$ and thus

$$\left| \frac{w_r - x_0}{r} \right| = 1.$$

Next, take $r \downarrow 0$ in the first inequality to get, since $|w_r - x_0| = r$,

$$|p| \leq \left\langle p, \lim_{r \downarrow 0} \frac{w_r - x_0}{r} \right\rangle \leq |p| \cos \alpha,$$

where α is the angle formed by p and $\lim_{r \downarrow 0} \frac{w_r - x_0}{r}$, which is then $\alpha = 0$. It follows that

$$\lim_{r \downarrow 0} \frac{w_r - x_0}{r} = \frac{p}{|p|}, \quad p \neq 0.$$

To conclude this part, pass to the limit as $r \downarrow 0$ in the extremes inequality in (17) to obtain, after dividing by r,

$$0 \leq \frac{1}{2} \left\langle X \frac{p}{|p|}, \frac{p}{|p|} \right\rangle |p| \;\Leftrightarrow\; 0 \leq \langle Xp, p \rangle = \Delta_\infty u(x_0).$$

In the general case, let $\hat{x} \in U$ and $\varphi \in C^2(U)$ be such that $u - \varphi$ has a local maximum at $\hat{x}$. Then, for y, w close to $\hat{x}$,

$$\varphi(\hat{x}) - \varphi(y) \leq u(\hat{x}) - u(y)$$

and

$$u(w) - u(\hat{x}) \leq \varphi(w) - \varphi(\hat{x}).$$

Then

$$\begin{aligned} \varphi(\hat{x}) - \varphi(y) &\leq u(\hat{x}) - u(y) \\ &\leq \max_{w \in \partial B_r(y)} (u(w) - u(\hat{x})) \frac{|\hat{x} - y|}{r - |\hat{x} - y|} \\ &\leq \max_{w \in \partial B_r(y)} (\varphi(w) - \varphi(\hat{x})) \frac{|\hat{x} - y|}{r - |\hat{x} - y|} \end{aligned}$$

and we have obtained (15) for the twice continuously differentiable function φ. Repeating the above reasoning, we conclude that

$$\Delta_\infty \varphi(\hat{x}) \geq 0$$

and the proof is complete. □

Entirely analogous results hold replacing $\infty-$subharmonic with $\infty-$superharmonic and comparison with cones from above with comparison with cones from below. We thus obtain the following result.

Theorem 5. *A function $u \in C(U)$ is $\infty-$harmonic if, and only if, it enjoys comparison with cones.*

6. Regularity

We now turn to regularity. For an open set U and $x \in U$, we introduce the notation

$$d(x) := \operatorname{dist}(x, \partial U).$$

Our first result is a Harnack inequality.

Lemma 4 (Harnack Inequality)**.** *Let $0 \geq u \in C(U)$ satisfy*

$$u(x) \leq u(y) + \max_{w \in \partial B_r(y)} \left(\frac{u(w) - u(y)}{r} \right) |x - y|, \tag{18}$$

for $x \in B_r(y) \subset\subset U$.

If $z \in U$ and $R < d(z)/4$, then

$$\sup_{B_R(z)} u \leq \frac{1}{3} \inf_{B_R(z)} u. \tag{19}$$

Proof. Take arbitrary $x, y \in B_R(z)$. Then (18) holds for r sufficiently large. Let $r \uparrow d(y)$ to get, using the fact that $u \leq 0$,

$$u(x) \leq u(y) \left(1 - \frac{|x - y|}{d(y)} \right). \tag{20}$$

We have

$$d(y) \geq 3R \qquad \text{and} \qquad |x - y| \leq 2R$$

and thus, from (20), we obtain

$$u(x) \leq u(y) \left(1 - \frac{2R}{3R} \right) = \frac{1}{3} u(y)$$

and the result follows. □

We now sharpen the estimate, with a direct proof of the result in [6], where, alternatively, the proof follows from looking at the $\infty-$Laplace equation as the limit as $p \to \infty$ of the $p-$Laplace equation.

Theorem 6 (The Harnack Inequality of Lindqvist–Manfredi)**.** *Let* $0 \geq u \in C(U)$ *satisfy* (18)*. If* $z \in U$ *and* $0 < R < d(z)$*, then*

$$u(x) \leq \exp\left(-\frac{|x-y|}{d(z)-R}\right) u(y), \quad \forall x, y \in B_R(z). \tag{21}$$

Proof. Let $x, y \in B_R(z)$, $m \in \mathbb{N}$ and define

$$x_k = x + k\,\frac{y-x}{m}, \quad k = 0, 1, \ldots, m.$$

We have, for every k,

$$|x_{k+1} - x_k| = \frac{|x-y|}{m} < d(x_{k+1}),$$

for m large enough, and

$$d(x_{k+1}) \geq d(z) - R.$$

We can then apply (20), with $x = x_k$ and $y = x_{k+1}$, to get

$$\begin{aligned} u(x_k) &\leq u(x_{k+1}) \left(1 - \frac{|x_{k+1} - x_k|}{d(x_{k+1})}\right) \\ &\leq u(x_{k+1}) \left(1 - \frac{|x-y|}{m(d(z)-R)}\right). \end{aligned}$$

Iterating, we obtain

$$u(x) = u(x_0) \leq u(y) \left(1 - \frac{|x-y|}{m(d(z)-R)}\right)^m,$$

and taking $m \to \infty$, we arrive at (21). □

This is indeed a sharper Harnack inequality when compared with (19). For starters, it is valid for every $R < d(z)$. Moreover, the constant is also better: taking $R = d(z)/4$, we obtain

$$\sup_{B_R(z)} u(x) \leq \exp\left(-\frac{d(z)/2}{d(z) - d(z)/4}\right) \inf_{B_R(z)} u(y) = \exp\left(-\frac{2}{3}\right) \inf_{B_R(z)} u(y)$$

and $\exp\left(-\frac{2}{3}\right) \approx 0.5134 > 0.3333 \approx \frac{1}{3}$.

The local Lipschitz regularity for $\infty-$harmonic functions is now a consequence of the Harnack inequality.

Theorem 7. *If* $u \in C(U)$ *is* $\infty-$*harmonic then it is locally Lipschitz and hence differentiable almost everywhere.*

Proof. We know u satisfies (18), since it enjoys comparison with cones from above. Take $z \in U$, $R < d(z)/4$ and $x, y \in B_R(z)$. Assume first that $u \leq 0$. Then (20) and the Harnack inequality (19) hold, and we get

$$\begin{aligned} u(x) - u(y) &\leq -u(y)\frac{|x-y|}{d(y)} \\ &\leq -\inf_{B_R(z)} u \, \frac{|x-y|}{3R} \\ &\leq -\sup_{B_R(z)} u \, \frac{|x-y|}{R}. \end{aligned}$$

If u is not non-positive, then this holds with u replaced by

$$v = u - \sup_{B_{4R}(z)} u \leq 0,$$

since $v = u + \mathit{const}$ still enjoys comparison with cones from above. We thus obtain

$$\begin{aligned} u(x) - u(y) = v(x) - v(y) &\leq -\sup_{B_R(z)} v \, \frac{|x-y|}{R} \\ &= \left(\sup_{B_{4R}(z)} u - \sup_{B_R(z)} u \right) \frac{|x-y|}{R} \end{aligned}$$

and, interchanging x and y,

$$|u(x) - u(y)| \leq \frac{1}{R} \left(\sup_{B_{4R}(z)} u - \sup_{B_R(z)} u \right) |x-y|.$$

□

The best regularity result to date is due to Evans–Smart [5] and asserts that $\infty-$harmonic functions are differentiable *everywhere*. It remains an outstanding open problem to prove the C^1 or $C^{1,\alpha}$ regularity, which are known to hold only in two-dimensions after the breakthroughs of Savin [9] and Evans–Savin [4].

In the recent contribution [8] it is shown that the solution of the obstacle problem for the $\infty-$Laplacian leaves a regular obstacle exactly as a $C^{1,\frac{1}{3}}-$function.

7. Existence

It is now time to deal with existence. We will need the following result; a proof is in [2].

Lemma 5. *Let $\mathcal{F} \subset C(U)$ be a family of functions that enjoy comparison with cones from above in U. Suppose*

$$h(x) = \sup_{v \in \mathcal{F}} v(x)$$

is finite and locally bounded above in U. Then $h \in C(U)$, and it enjoys comparison with cones from above in U.

The existence result we present holds for U unbounded if the boundary function f is allowed to grow at most linearly at infinity. Note that it settles also the existence for problem (3), since u is $\infty-$harmonic in U if, and only if, $u \in AML(U)$.

Theorem 8. *Let $U \subset \mathbb{R}^n$ be open, $0 \in \partial U$ and $f \in C(\partial U)$. Let $A^{\pm}, B^{\pm} \in \mathbb{R}$, $A^+ \geq A^-$ and*

$$A^-|x| + B^- \leq f(x) \leq A^+|x| + B^+, \quad \forall x \in \partial U. \tag{22}$$

There exists $u \in C(\overline{U})$ which is $\infty-$harmonic in U and satisfies $u = f$ on ∂U. Moreover,

$$A^-|x| + B^- \leq u(x) \leq A^+|x| + B^+, \quad \forall x \in \overline{U}. \tag{23}$$

The proof is an application of Perron's method. By translation, we can always assume $0 \in \partial U$ so this assumption is not restrictive and it is used to simplify the notation.

We start by defining two functions $\underline{h}, \overline{h} : \mathbb{R}^n \to \mathbb{R}$ by

$$\underline{h}(x) = \sup\{\underline{C}(x) \, : \, \underline{C}(x) = a|x-z| + b, a < A^-, z \in \partial U, \underline{C} \leq f \text{ on } \partial U\}$$

and

$$\overline{h}(x) = \inf\{\overline{C}(x) \, : \, \overline{C}(x) = a|x-z| + b, a > A^+, z \in \partial U, \overline{C} \geq f \text{ on } \partial U\}$$

with the properties stated in the next lemma.

Lemma 6. *The functions $\underline{h}$ and $\overline{h}$ are well defined and continuous. Moreover,*

$$A^-|x| + B^- \leq \underline{h}(x) \leq \overline{h}(x) \leq A^+|x| + B^+, \quad \forall x \in \mathbb{R}^n, \tag{24}$$

$$\underline{h} = \overline{h} = f \text{ on } \partial U, \tag{25}$$

$\underline{h}$ enjoys comparison with cones from above and $\overline{h}$ enjoys comparison with cones from below.

Proof. We argue for $\overline{h}$; analogous arguments hold for $\underline{h}$. First observe that any cone in the family that is used to define $\overline{h}$ is bounded below by the cone

$$A^+|x-z|+f(z),$$

and then so is $\overline{h}$. The question is to show that the family is non-empty. Since $0\in\partial U$, we may take

$$\overline{C}(x)=(A^++\epsilon)|x|+B^+,\quad \epsilon>0$$

and so $\overline{h}$ is well defined. This also readily implies that $\overline{h}(x)\le A^+|x|+B^+$.

To show that (25) holds for $\overline{h}$, fix $0\neq z\in\partial U$ and $\epsilon>0$. By the continuity of f, there exists $\delta>0$ such that

$$f(x)<f(z)+\epsilon,\quad \forall x\in B_\delta(z)\cap\partial U. \tag{26}$$

Then choose $a>\max\{A^+,0\}$ such that

$$f(z)+\epsilon+a\delta>\max_{\overline{B_\delta}(z)}(A^+|x|+B^+) \tag{27}$$

and

$$f(z)+\epsilon+a|z|>B^+. \tag{28}$$

Define the cones

$$\overline{C}(x):=a|x-z|+f(z)+\epsilon$$

and

$$C^+(x):=A^+|x|+B^+,$$

and the open set

$$W:=\left\{x\in\mathbb{R}^n\setminus\overline{B_\delta}(z)\ :\ \overline{C}(x)<C^+(x)\right\}.$$

Since $a>A^+$, W is bounded:

$$\lim_{|x|\to\infty}\big(a|x-z|-A^+|x|\big)=+\infty.$$

Moreover, by (27), $\partial B_\delta(z)\cap\overline{W}=\emptyset$, and then $\overline{C}=C^+$ on ∂W. Since both vertices of the cones, 0 and z, do not belong to W (to see that $0\notin W$, use (28)), we conclude, by Corollary 1 and the reasoning at the end of the proof of Theorem 2, that $\overline{C}=C^+$ also in W. Thus $W=\emptyset$ and

$$\overline{C}(x)\ge C^+(x),\quad \forall x\in\mathbb{R}^n\setminus B_\delta(z).$$

This and (26) implies that

$$\overline{C}\ge f\ \text{ on }\ \partial U.$$

Now, $\overline{C}(z)=f(z)+\epsilon$ and so $\overline{h}(z)=f(z)$ and (25) holds for $\overline{h}$.

To prove that $\underline{h} \le \overline{h}$, take any two cones

$$\underline{C}(x) = \underline{a}|x - \underline{z}| + \underline{b} \qquad \text{and} \qquad \overline{C}(x) = \overline{a}|x - \overline{z}| + \overline{b}$$

entering in the definition of $\underline{h}$ and $\overline{h}$, respectively. Since

$$\underline{C} \le f \le \overline{C} \ \text{ on } \partial U,$$

$$\underline{z}, \overline{z} \in \partial U$$

and

$$\overline{a} > A^+ \ge A^- > \underline{a},$$

the set where $\underline{C} > \overline{C}$ is bounded, contains neither vertex and the two cones agree on its boundary. Arguing as before, we conclude the set is empty and so $\underline{C} \le \overline{C}$, which implies

$$\underline{h} \le \overline{h} \ \text{ in } R^n$$

and (24) is proved. In particular, $\underline{h}$ and $\overline{h}$ are locally bounded and comparison with cones (respectively, from above and from below) follows from Lemma 5 and its variant.

We are left to prove the continuity of $\underline{h}$ and $\overline{h}$. First observe that $\underline{h}$ is lower semicontinuous (as the supremum of continuous functions) and $\overline{h}$ is upper semicontinuous (as the infimum of continuous functions) in R^n. The continuity in U also follows from Lemma 5. Since $\underline{h}$ enjoys comparison with cones from above, the idea is to use the Harnack Inequality (Lemma 4, which holds for lower semicontinuous functions) as in the proof of Theorem 7.

To prove the continuity of $\underline{h}$ on ∂U, use (25) and (24) to get

$$f(x) \le \liminf_{y\to x} \underline{h}(y) \le \limsup_{y\to x} \underline{h}(y) \le \limsup_{y\to x} \overline{h}(y) \le f(x), \quad x \in \partial U.$$

The case of $\overline{h}$ is treated analogously. □

We need yet another lemma.

Lemma 7. *Suppose $u \in C(U)$ enjoys comparison with cones from above in U but does not enjoy comparison with cones from below in U. Then, there exists a nonempty set $W \subset\subset U$ and a cone $C(x) = a|x-z|+b$, with $z \notin W$, such that $u = C$ on ∂W, $u < C$ on W and the function $\hat{u}$ defined by*

$$\hat{u} = u \ \text{ in } U \setminus W \qquad \text{and} \qquad \hat{u} = C \ \text{ in } W \tag{29}$$

enjoys comparison with cones from above in U. Moreover, if u is Lipschitz in U, then so is $\hat{u}$ and

$$\mathrm{Lip}_{\hat{u}}(U) \le \mathrm{Lip}_u(U).$$

Proof. That there exist W and C satisfying the conditions of the lemma follows from the proof of the necessity in Theorem 2, more correctly, from its variant corresponding to comparison with cones from below.

Let's show that $\hat{u}$ defined by (29) enjoys comparison with cones from above in U. Suppose not; then, again from the proof of Theorem 2, there exists a nonempty set $\tilde{W} \subset\subset U$ and a cone $\tilde{C}(x) = \tilde{a}|x - \tilde{z}| + \tilde{b}$, with $\tilde{z} \notin \tilde{W}$, such that $\hat{u} = \tilde{C}$ on $\partial\tilde{W}$ and $\hat{u} > \tilde{C}$ in $\tilde{W}$. Since u enjoys comparison with cones from above in U and $u \leq \hat{u} = \tilde{C}$ on $\partial\tilde{W}$, we have $u \leq \tilde{C}$ also in $\tilde{W}$. This implies that $\tilde{W} \subset W$ because

$$u \leq \tilde{C} < \hat{u} \;\text{ in } \tilde{W} \qquad \text{and} \qquad u = \hat{u} \;\text{ in } U \setminus W.$$

Thus, on $\partial\tilde{W} \subset W \cup \partial W$,

$$\tilde{C} = \hat{u} = C.$$

Since the vertices of the cones C and $\tilde{C}$ are outside $\tilde{W}$, this implies

$$\tilde{C} \equiv C \equiv \hat{u} \;\text{ in } \tilde{W}$$

and so $\tilde{W} = \emptyset$, a contradiction.

Finally, since

$$\mathrm{Lip}_{\hat{u}}(W) = \mathrm{Lip}_C(W) = \mathrm{Lip}_C(\partial W) = \mathrm{Lip}_u(\partial W) \leq \mathrm{Lip}_u(U)$$

(note that the vertex of C is outside W), we conclude that

$$\mathrm{Lip}_{\hat{u}}(U) = \max\{\mathrm{Lip}_{\hat{u}}(W), \mathrm{Lip}_{\hat{u}}(U \setminus W)\} \leq \mathrm{Lip}_u(U).$$

□

We are now ready to prove Theorem 8.

Proof. Define

$$u(x) := \sup\Big\{v(x) \,:\, \underline{h} \leq v \leq \overline{h} \;\text{ and }\; v \in CCA(U)\Big\}, \quad x \in \overline{U},$$

where, by $v \in CCA(U)$ we mean that v enjoys comparison with cones from above in U.

By Lemma 6, the set includes $\underline{h}$ so it is not empty and u is well defined; it follows from Lemma 5 that it enjoys comparison with cones from above in U, and, from (25), that $u \in C(\overline{U})$ and $u = f$ on ∂U.

If u enjoys comparison with cones then u is $\infty-$harmonic and the proof is complete. Otherwise, u does not enjoy comparison with cones from below and, by Lemma 7, there exists a nonempty set $W \subset\subset U$ and a cone C, with vertex outside W, such that

$$u = C \;\text{ on } \partial W \qquad \text{and} \qquad u < C \;\text{ in } W,$$

and a continuous function $\hat{u}$, enjoying comparison with cones from above, such that

$$\hat{u} = u \quad \text{in } U \setminus W \qquad \text{and} \qquad \hat{u} = C \quad \text{in } W.$$

It is obvious that $\underline{h} \le u \le \hat{u}$. We claim that also $\hat{u} \le \overline{h}$ in U, which then contradicts the definition of u.

Since $\overline{h}$ enjoys comparison with cones from below and

$$\overline{h} \ge C = u \quad \text{on } \partial W,$$

we have $\overline{h} \ge C = \hat{u}$ also in W. Since in $U \setminus W$, $\hat{u} = u \le \overline{h}$, the proof is complete. □

8. Uniqueness

The uniqueness reveals the extent to which the notion of viscosity solution is the appropriate one to deal with the $\infty-$Laplace equation. Given any pde, we can, of course, define any reasonable notion of solution; what makes the difference is that, for that notion, not only existence but also uniqueness holds.

The question of the uniqueness of $\infty-$harmonic functions remained open for more than two decades, before it was settled by Jensen in [7] using the full machinery of viscosity solutions. The proof we will next present is much simpler and exploits the equivalence between being $\infty-$harmonic and enjoying comparison with cones. It is a surprisingly easy and beautiful proof due to Armstrong and Smart [1].

We start with some notation. Given an open and bounded subset $U \subset \mathbb{R}^n$ and $r > 0$, let

$$U_r := \left\{ x \in U \; : \; \overline{B_r}(x) \subset U \right\}.$$

For $u \in C(U)$ and $x \in U_r$, define

$$u^r(x) := \max_{\overline{B_r}(x)} u \qquad \text{and} \qquad u_r(x) := \min_{\overline{B_r}(x)} u,$$

and let

$$S_r^+ u(x) = \frac{u^r(x) - u(x)}{r} \qquad \text{and} \qquad S_r^- u(x) = \frac{u(x) - u_r(x)}{r}.$$

Note that both $S_r^+ u \ge 0$ and $S_r^- u \ge 0$.

The first result we prove is a comparison principle at the discrete level, for the finite difference equation $S_r^- u = S_r^+ u$.

Lemma 8. *Assume $u, v \in C(U) \cap L^\infty(U)$ satisfy*

$$S_r^- u(x) - S_r^+ u(x) \leq 0 \leq S_r^- v(x) - S_r^+ v(x), \ \forall x \in U_r. \tag{30}$$

Then

$$\sup_U (u - v) = \sup_{U \setminus U_r} (u - v).$$

Proof. Suppose the thesis does not hold, *i.e.*,

$$\sup_U (u - v) > \sup_{U \setminus U_r} (u - v).$$

The set

$$E := \left\{ x \in U \,:\, (u - v)(x) = \sup_U (u - v) \right\}$$

is then nonempty, closed and contained in U_r. Define

$$F := \left\{ x \in E \,:\, u(x) = \max_E u \right\},$$

which is also nonempty and closed, and select a point $x_0 \in \partial F$. Since $u - v$ attains its maximum at x_0 (because $x_0 \in \overline{F} = F \subset E$), we have

$$S_r^- v(x_0) \leq S_r^- u(x_0) \ \Leftrightarrow \ u_r(x_0) - v_r(x_0) \leq (u - v)(x_0), \tag{31}$$

which holds since $\max(f - g) \geq \min f - \min g$.[1]

We now consider two cases.

(1) $S_r^+ u(x_0) = 0$: from (30), we get

$$S_r^- u(x_0) \leq 0 \ \Rightarrow \ S_r^- u(x_0) = 0$$

and, from (31),

$$S_r^- v(x_0) \leq 0 \ \Rightarrow \ S_r^- v(x_0) = 0.$$

Using the other inequality in (30),

$$0 \leq 0 - S_r^+ v(x_0) \ \Rightarrow \ S_r^+ v(x_0) = 0.$$

So

$$\max_{\overline{B_r}(x_0)} u = u(x_0) = \min_{\overline{B_r}(x_0)} u$$

and

$$\max_{\overline{B_r}(x_0)} v = v(x_0) = \min_{\overline{B_r}(x_0)} v,$$

[1] In fact, $\max(f - g) = \max_x (f(x) - g(x)) \geq \max_x (\min f - g(x)) = \min f - \min g$.

and both u and v are constant in $\overline{B_r}(x_0)$. Thus $B_r(x_0) \subset F$; in fact, if $y \in B_r(x_0)$ then, since $x_0 \in E$,

$$(u-v)(y) = (u-v)(x_0) = \sup_U (u-v) \ \Rightarrow \ y \in E;$$

but also

$$u(y) = u(x_0) = \max_E u,$$

since $x_0 \in F$; thus $y \in F$. We conclude that $x_0 \in int(F)$ and so $x_0 \notin \partial F$, a contradiction.

(2) $S_r^+ u(x_0) > 0$: select a point $z \in \overline{B_r}(x_0)$ such that

$$rS_r^+ u(x_0) = u(z) - u(x_0).$$

Since $u(z) > u(x_0)$ and $x_0 \in F$, we see that $z \notin E$. From this, we deduce that

$$rS_r^+ v(x_0) \geq v(z) - v(x_0) > u(z) - u(x_0) = rS_r^+ u(x_0). \tag{32}$$

To justify the strict inequality above, observe that

$$(u-v)(z) \leq (u-v)(x_0) = \sup_U (u-v),$$

because $x_0 \in F \subset E$, and equality does not hold since then $z \in E$. Finally, combining (31) and (32), we get

$$S_r^- v(x_0) - S_r^+ v(x_0) < S_r^- u(x_0) - S_r^+ u(x_0),$$

which contradicts (30).

□

The next result establishes a link between the continuous and the discrete levels, showing that solutions of the pde can be suitably modified in order to solve the finite difference equation.

Lemma 9. *If $u \in C(U)$ is $\infty-$subharmonic in U, then*

$$S_r^- u^r(x) - S_r^+ u^r(x) \leq 0, \quad \forall x \in U_{2r},$$

and if $v \in C(U)$ is $\infty-$superharmonic in U, then

$$S_r^- v_r(x) - S_r^+ v_r(x) \geq 0, \quad \forall x \in U_{2r}.$$

Proof. We just prove the first statement; the second one follows from the fact that $(-v)^r = -v_r$.

Fix a point $x_0 \in U_{2r}$. Select $y_0 \in \overline{B_r}(x_0)$ and $z_0 \in \overline{B_{2r}}(x_0)$ such that

$$u(y_0) = u^r(x_0) \qquad \textit{and} \qquad u(z_0) = u^{2r}(x_0).$$

Then,

$$\begin{aligned} r\left[S_r^- u^r(x_0) - S_r^+ u^r(x_0)\right] &= 2u^r(x_0) - (u^r)_r(x_0) - (u^r)^r(x_0) \\ &\leq 2u^r(x_0) - u^{2r}(x_0) - u(x_0) \\ &= 2u(y_0) - u(z_0) - u(x_0). \end{aligned}$$

We next justify why the inequality holds.

(1) $(u^r)^r(x) = u^{2r}(x)$: we have

$$(u^r)^r(x) = \max_{z\in\overline{B_r}(x)} u^r(z) = \max_{z\in\overline{B_r}(x)} \max_{y\in\overline{B_r}(z)} u(y)$$

and

$$u^{2r}(x) = \max_{y\in\overline{B_{2r}}(x)} u(y).$$

• If $z \in \overline{B_r}(x)$ and $y \in \overline{B_r}(z)$ then $y \in \overline{B_{2r}}(x)$. In fact,

$$|z - x| \leq r \ \wedge \ |y - z| \leq r$$

$$\Rightarrow \ |y - x| \leq |y - z| + |z - x| \leq 2r$$

and thus

$$(u^r)^r(x) \leq u^{2r}(x).$$

• If $y \in \overline{B_{2r}}(x)$ then $y \in \overline{B_r}(z)$, for a certain $z \in \overline{B_r}(x)$; just take z to be the middle point of the segment $[x, y]$. So, also

$$u^{2r}(x) \leq (u^r)^r(x).$$

(2) $(u^r)_r(x) \geq u(x)$: we have

$$(u^r)_r(x) = \min_{z\in\overline{B_r}(x)} u^r(z) = \min_{z\in\overline{B_r}(x)} \max_{y\in\overline{B_r}(z)} u(y).$$

Since

$$\max_{y\in\overline{B_r}(z)} u(y) \geq u(x), \ \forall z \in \overline{B_r}(x),$$

the result follows.

Now, clearly,

$$u(w) \leq u(x_0) + \frac{u(z_0) - u(x_0)}{2r}|w - x_0|, \quad \forall w \in \partial\left(B_{2r}(x_0) \setminus \{x_0\}\right).$$

Since u enjoys comparison with cones from above, because u is $\infty-$subharmonic, the inequality also holds for every $w \in B_{2r}(x_0) \setminus \{x_0\}$ and, since it holds trivially for $w = x_0$, for every $w \in \overline{B_{2r}}(x_0)$.

Putting $w = y_0$ and using the fact that $|y_0 - x_0| \le r$, we get

$$\begin{aligned} u(y_0) &\le u(x_0) + \frac{u(z_0) - u(x_0)}{2r}|y_0 - x_0| \\ &\le u(x_0) + \frac{u(z_0) - u(x_0)}{2} \end{aligned}$$

and then

$$2u(y_0) - u(x_0) - u(z_0) \le 0,$$

and the proof is complete. □

Theorem 9 (Jensen's Uniqueness Theorem)**.** *Let $u, v \in C(\overline{U})$ be, respectively, $\infty-$subharmonic and $\infty-$superharmonic. Then*

$$\max_{\overline{U}}(u - v) = \max_{\partial U}(u - v).$$

Proof. From Lemmas 9 and 8,

$$\sup_{U_r}(u^r - v_r) = \sup_{U_r \setminus U_{2r}}(u^r - v_r), \quad \forall r > 0.$$

To get the result, let $r \downarrow 0$ and use the local uniform converge of u^r and v_r to u and v, respectively. □

Problems

(1) Let $u \in C(\overline{U})$. Show that $\text{Lip}_u(U) = \text{Lip}_u(\overline{U})$.

(2) Show that the infimum and the supremum of a family of Lipschitz functions, with a fixed Lipschitz constant, is Lipschitz and has, if it is finite, the same Lipschitz constant.

(3) Let $n = 1$ and $U = (-2, -1) \cup (1, 2)$. Consider $f : \partial U \to \mathbb{R}$ defined by $f(-2) = 0$, $f(-1) = 1$ and $f(1) = f(2) = 1$.

(a) Determine $\text{Lip}_f(\partial U)$.

(b) Compute the MacShane-Whitney extensions of f to U.

(c) Choose the extension of f which is in $\text{AML}(U)$.

(4) Consider the modulus function $u(x) = |x|$ in $\mathbb{R}^n$.

(a) Prove u is $\infty-$subharmonic.

(b) Give a short justification to the fact that it is not $\infty-$harmonic.

(c) Use the definition to show the previous fact.

(5) Let $n = 2$, $u(x) = |x|$ and $v(x) = x_1$.

(a) Construct a bounded set $U \subset \mathbb{R}^2 \setminus \{0\}$ such that $v < u$ on ∂U except at two points and $u = v$ on the line segment joining these two points.

(b) Conclude there is no strong comparison principle for $\infty-$harmonic functions.

(6) Show that a function $u \in C(U)$ is $\infty-$subharmonic if, and only if, the map

$$r \to u^r(x) = \max_{\overline{B_r(x)}} u$$

is convex on $[0, d(x))$, for every $x \in U$.

(7) (Liouville's Theorem) Prove that if u is $\infty-$harmonic in $\mathbb{R}^n$ and u is bounded below, then u is constant.

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