A VERY SHORT INTRODUCTION TO REGULARITY THEORY FOR LINEAR ELLIPTIC EQUATIONS

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Abstract. These lecture notes provide a short introduction to regularity theory for second-order linear elliptic PDEs. The material covered ranges from equations with regular coefficients to the case of coefficients that are merely measurable and bounded. In the first part, where I essentially follow the book of L.C. Evans [2], the difference quotient method of Nirenberg is developed. In the second part, I present the basics of the De Giorgi-Nash-Moser theory, mainly based on [3, 4, 10].

Contents

1. Linear elliptic equations

Let us consider a bounded domain (connected and open) $U \subset \mathbb{R}^n$ and focus our attention on the partial differential equation (PDE)

$$\mathcal{A}u := -\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i}\right)_{x_j} = f \quad \text{in } U. \tag{1.1}$$

The unknown u, as well as the coefficients a^{ij} and the right-hand side f, are functions of $x \in U$. The notation for derivatives is standard: $u_{x_i} = \partial u / \partial x_i$. The equation is the archetypal second-order linear PDE in divergence form. By this we mean:

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linear: the differential operator $\mathcal{A}$ is linear;

of second-order: the highest derivate of the unknown u involved in the equation is the second;

in divergence form: $\mathcal{A}$ can be written as div$\vec{F}$, with $\vec{F} = (F_j)$ and

$$F_j = -\sum_{i=1}^{n} a^{ij} u_{x_i}, \quad j = 1 \ldots, n.$$

We complete our set of structural assumptions on the PDE by requiring the differential operator $\mathcal{A}$ to be uniformly elliptic and symmetric:

uniform ellipticity: there is a constant $\lambda > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij} \xi_i \xi_j \geq \lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \quad \text{a.e. in } U; \tag{1.2}$$

symmetry: $a^{ij} = a^{ji}$, a.e. in U $(i, j = 1, \ldots, n)$.

This means that, a.e. in U, the symmetric matrix $[a^{ij}]$ is positive-definite, with its smallest eigenvalue greater than or equal to λ.

From now on, we use the convention of summing over repeated indices. We start by rigorously defining what we mean by a weak solution.

Definition 1. *A function $u \in H^1(U)$ is a weak solution of* (1.1) *if*

$$\int_U a^{ij} u_{x_i} \varphi_{x_j} \, dx = \int_U f \varphi \, dx, \quad \forall \varphi \in H_0^1(U). \tag{1.3}$$

Before moving on, we recall an elementary inequality that will be used extensively later.

Lemma 1 (Cauchy's inequality)**.** *Let $\epsilon > 0$. Then*

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}, \quad \forall a, b > 0. \tag{1.4}$$

Proof. Just observe that $(2\epsilon a - b)^2 \geq 0$. $\square$

2. Smooth coefficients

2.1. **Difference quotients.** The technical tools put forward here will be instrumental in the sequel. By $V \Subset U$ we mean V is compactly contained in U, that is, the closure of V is a compact subset of U.

Definition 2. *Let $u \in L^1_{\mathrm{loc}}(U)$, $V \Subset U$, $0 < |h| < \mathrm{dist}(V, \partial U)$ and $k \in \{1, \dots, n\}$. The difference quotient of u in V, in the k^{th}–direction and with size h, is*

$$D^h_k u(x) = \frac{u^h_k(x) - u(x)}{h}, \qquad x \in V,$$

where $u^h_k(x) := u(x + he_k)$. We write $D^h u := (D^h_1 u, \dots, D^h_n u)$.

We start with two formulas: the product rule for difference quotients and a sort of integration-by-parts formula.

Lemma 2. *Let $v, w \in L^1_{\mathrm{loc}}(U)$, $V \Subset U$, $0 < |h| < \mathrm{dist}(V, \partial U)$ and $k \in \{1, \dots, n\}$. Then*

$$D^h_k(vw) = v D^h_k w + w^h_k D^h_k v, \quad \text{in } V. \tag{2.1}$$

Proof. The proof is just algebra:

$$\begin{aligned} D^h_k(vw)(x) &= \frac{v^h_k(x) w^h_k(x) - v(x)w(x)}{h} \\ &= v(x)\frac{w^h_k(x) - w(x)}{h} + w^h_k(x)\frac{v^h_k(x) - v(x)}{h} \\ &= v(x) D^h_k w(x) + w^h_k(x) D^h_k v(x). \end{aligned}$$

□

Remark 1. *Observe that we also have $D^h_k(vw) = v^h_k D^h_k w + w D^h_k v$ and so, as expected, $D^h_k(vw) = D^h_k(wv)$.*

Lemma 3. *Let $v, w \in L^2_{\mathrm{loc}}(U)$ and assume $W := \operatorname{Supp} w \subset U$. Take $0 < |h| < \mathrm{dist}(W, \partial U)$ and $k \in \{1, \dots, n\}$. Then*

$$\int_U v D^{-h}_k w \, dx = - \int_U w D^h_k v \, dx. \tag{2.2}$$

Proof. Changing variables,

$$\begin{aligned}
\int_U vD_k^{-h}w\,dx &= \int_U v(x)\frac{w_k^{-h}(x)-w(x)}{-h}\,dx \\
&= -\frac{1}{h}\left(\int_{W+he_k} v(x)w_k^{-h}(x)dx - \int_W v(x)w(x)dx\right) \\
&= -\frac{1}{h}\left(\int_W v_k^h(y)w(y)dy - \int_W v(x)w(x)dx\right) \\
&= -\frac{1}{h}\left(\int_W v_k^h(x)w(x)dx - \int_W v(x)w(x)dx\right) \\
&= -\int_W w(x)\frac{v_k^h(x)-v(x)}{h}\,dx \\
&= -\int_U wD_k^h v\,dx.
\end{aligned}$$

□

We now prove the crucial relation between difference quotients and weak derivatives.

Theorem 1. *Let $u \in H^1(U)$. Then, for each $V \Subset U$,*

$$\left\|D^h u\right\|_{L^2(V)} \leq C\,\|Du\|_{L^2(U)}\,,$$

for some constant C and all $0 < |h| < \frac{1}{2}\mathrm{dist}(V, \partial U)$.

Proof. We assume u is smooth; the general case follows by approximation. For each $x \in V$, $i = 1, \ldots, n$ and $0 < |h| < \frac{1}{2}\mathrm{dist}(V, \partial U)$, we have

$$u(x+he_i) - u(x) = h\int_0^1 u_{x_i}(x+the_i)\,dt$$

so that

$$|D_i^h u(x)| \leq \int_0^1 |Du(x+the_i)|\,dt.$$

Then

$$\begin{aligned}
\int_V \left|D^h u\right|^2 dx &\leq \sum_{i=1}^n \int_V \int_0^1 |Du(x+the_i)|^2\,dtdx \\
&= \sum_{i=1}^n \int_0^1 \int_V |Du(x+the_i)|^2\,dxdt \\
&\leq n\int_U |Du|^2\,dx.
\end{aligned}$$

□

Theorem 2. *Let $V \Subset U$ and $u \in L^2(V)$. If there exists a constant C such that*

$$\left\| D^h u \right\|_{L^2(V)} \leq C, \qquad \forall 0 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial U)$$

then

$$u \in H^1(V), \quad \textit{with } \|Du\|_{L^2(V)} \leq C.$$

Proof. Choose $i \in \{1, \ldots, n\}$. The assumption of the theorem implies that

$$\sup_{0<|h|<\frac{1}{2}\operatorname{dist}(V,\partial U)} \|D_i^{-h} u\|_{L^2(V)} \leq C.$$

Since L^2 is reflexive, there exists a function $v_i \in L^2(V)$, with $\|v_i\|_{L^2(V)} \leq C$, such that

$$D_i^{-h_k} u \rightharpoonup v_i \qquad \text{weakly in } L^2(V),$$

for a subsequence $h_k \to 0$.

Now let $\phi \in C_c^\infty(V)$; we have, using (2.2),

$$\begin{aligned}
\int_V u\phi_{x_i} dx &= \int_U u\phi_{x_i} dx \\
&= \lim_{h_k \to 0} \int_U u D_i^{h_k} \phi \, dx \\
&= -\lim_{h_k \to 0} \int_V \phi D_i^{-h_k} u \, dx \\
&= -\int_V \phi v_i \, dx \\
&= -\int_V v_i \phi \, dx,
\end{aligned}$$

which means that $v_i = u_{x_i}$ in the weak sense ($i = 1, \ldots, n$). We then have $Du \in L^2(V)$, with $\|Du\|_{L^2(V)} \leq C$, and thus $u \in H^1(V)$.

[As an exercise, show that the difference quotient converges uniformly to the derivative, so it is legitimate to interchange the limit with the integral in the calculations above.] □

2.2. **Interior regularity.**

Theorem 3. *Assume $a^{ij} \in C^1(U)$ and $f \in L^2(U)$, and let $u \in H^1(U)$ be a weak solution of* (1.1)*. Then $u \in H^2_{\text{loc}}(U)$ and, for each open subset $V \Subset U$, we have*

$$\|u\|_{H^2(V)} \leq C \left(\|u\|_{H^1(U)} + \|f\|_{L^2(U)} \right),$$

with the constant C depending only on V and the data.

Proof. Fix an arbitrary open set $V \Subset U$ and choose another open set W such that $V \Subset W \Subset U$. Then consider a cutoff function ϕ, that is, a smooth function $0 \leq \phi \leq 1$ such that

$$\phi \equiv 1 \text{ in } V \qquad \text{and} \qquad \phi \equiv 0 \text{ in } \mathbb{R}^n \setminus W.$$

Since *a priori*, we only know the derivatives of u are L^2 functions, we cannot use them to test the equation. But any difference quotient

$$D_k^h u(x) = \frac{u(x + he_k) - u(x)}{h},$$

with $k \in \{1, \ldots, n\}$ and $0 < |h| < \frac{1}{2}\mathrm{dist}(V, \partial U)$, has the same regularity of u and so

$$\varphi := -D_k^{-h}\left(\phi^2 D_k^h u\right)$$

is a legitimate test function in (1.3) since it belongs to $H_0^1(U)$ due to the cutoff. We thus get

$$\Upsilon := -\int_U a^{ij} u_{x_i} \left[D_k^{-h}\left(\phi^2 D_k^h u\right)\right]_{x_j} dx = -\int_U f\, D_k^{-h}\left(\phi^2 D_k^h u\right) dx =: \Theta.$$

We start with the left-hand side Υ. Using formulas (2.1) and (2.2), we obtain

$$\begin{aligned}
\Upsilon &= -\int_U a^{ij} u_{x_i} D_k^{-h}\left[\left(\phi^2 D_k^h u\right)_{x_j}\right] dx \\
&= \int_U D_k^h\left(a^{ij} u_{x_i}\right)\left(\phi^2 D_k^h u\right)_{x_j} dx \\
&= \int_U a^{ij} D_k^h u_{x_i}\left(\phi^2 D_k^h u\right)_{x_j} + \left(D_k^h a^{ij}\right)(u_{x_i})_k^h \left(\phi^2 D_k^h u\right)_{x_j} dx \\
&= \int_U a^{ij} D_k^h u_{x_i} D_k^h u_{x_j} \phi^2 dx \\
&\quad + \int_U \Big[a^{ij} D_k^h u_{x_i} D_k^h u\left(2\phi\phi_{x_j}\right) + \left(D_k^h a^{ij}\right)(u_k^h)_{x_i} D_k^h u_{x_j} \phi^2 \\
&\qquad\qquad + \left(D_k^h a^{ij}\right)(u_k^h)_{x_i} D_k^h u\left(2\phi\phi_{x_j}\right)\Big] dx \\
&= \Upsilon_1 + \Upsilon_2.
\end{aligned}$$

We estimate Υ_1 from below using the ellipticity condition. This is a crucial step in the proof, using the structural, algebraic assumption of ellipticity to obtain an analytical estimate:

$$\Upsilon_1 = \int_U a^{ij} D_k^h u_{x_i} D_k^h u_{x_j} \phi^2 dx \geq \lambda \int_U \left|D_k^h D u\right|^2 \phi^2 dx.$$

To estimate Υ_2 from above, we use the other crucial assumption, namely that the coefficients a^{ij} are regular (in our case, of class C^1):

$$|\Upsilon_2| \leq C \int_U \left|D_k^h Du\right| \left|D_k^h u\right| \phi + \left|D_k^h Du\right| |Du| \, \phi + \left|D_k^h u\right| |Du| \, \phi \, dx,$$

where the constant C only depends on dimension, the C^1-norm of the coefficients a^{ij} and the cutoff ϕ. Note that the difference quotients $D_k^h a^{ij}$ are uniformly bounded in U (i.e., the constant C does not depend on k nor on h) precisely because $a^{ij} \in C^1(U)$. Observe also the role of the cutoff function ϕ, in particular in rendering irrelevant the translations of the partial derivatives of u.

To absorb terms to the left-hand side, we now apply Cauchy's inequality (1.4), with $\epsilon = \lambda/4$ on the first two terms and $\epsilon = 1/2$ on the last, to obtain

$$\begin{aligned} |\Upsilon_2| &\leq \frac{\lambda}{2} \int_U \left|D_k^h Du\right|^2 \phi^2 dx + \left(\frac{C^2}{\lambda} + \frac{C}{2}\right) \int_W \left|D_k^h u\right|^2 + |Du|^2 \, dx \\ &\leq \frac{\lambda}{2} \int_U \left|D_k^h Du\right|^2 \phi^2 dx + C \int_U |Du|^2 \, dx, \end{aligned}$$

also using Theorem 1. Combining the different estimates, we finally get

$$\Upsilon = \Upsilon_1 + \Upsilon_2 \geq \frac{\lambda}{2} \int_U \left|D_k^h Du\right|^2 \phi^2 dx - C \int_U |Du|^2 \, dx.$$

We now shift our attention to the right-hand side Θ. Observe first that, due to Theorem 1 and the elementary inequality $(a+b)^2 \leq 2(a^2+b^2)$,

$$\begin{aligned} \int_U \left|D_k^{-h}\left(\phi^2 D_k^h u\right)\right|^2 dx &\leq \int_U \left|D\left(\phi^2 D_k^h u\right)\right|^2 dx \\ &\leq 2 \int_U \phi^4 \left|D_k^h Du\right|^2 dx + 8 \int_U \phi^2 |D\phi|^2 \left|D_k^h u\right|^2 dx \\ &\leq 2 \int_U \left|D_k^h Du\right|^2 \phi^2 dx + C \int_W \left|D_k^h u\right|^2 dx \\ &\leq C^* \int_U \left|D_k^h Du\right|^2 \phi^2 + |Du|^2 \, dx. \end{aligned}$$

Thus, using Cauchy's inequality (1.4), with $\epsilon = \lambda/4C^*$, and the previous estimate, we obtain

$$\begin{aligned} |\Theta| &\leq \int_U |f| \left|D_k^{-h}\left(\phi^2 D_k^h u\right)\right| dx \\ &\leq \frac{\lambda}{4C^*} \int_U \left|D_k^{-h}\left(\phi^2 D_k^h u\right)\right|^2 dx + \frac{C^*}{\lambda} \int_U |f|^2 \, dx \\ &\leq \frac{\lambda}{4} \int_U \left|D_k^h Du\right|^2 \phi^2 \, dx + C \int_U |Du|^2 + |f|^2 dx. \end{aligned}$$

Putting everything together, we finally get

$$\int_V \left|D_k^h Du\right|^2 dx \leq \int_U \left|D_k^h Du\right|^2 \phi^2 dx \leq C \int_U |Du|^2 + |f|^2 dx,$$

for all $k \in \{1, \ldots, n\}$ and all $0 < |h| < \frac{1}{2}\mathrm{dist}(V, \partial U)$. Applying Theorem 2, we conclude that $Du \in H^1_{\mathrm{loc}}(U)$, and thus $u \in H^2_{\mathrm{loc}}(U)$, with the estimate

$$\|u\|_{H^2(V)} \leq C \left(\|u\|_{H^1(U)} + \|f\|_{L^2(U)}\right).$$

$\square$

3. Bounded coefficients

3.1. **Hilbert's 19th problem.** In the academic year 1956-1957, John Nash held a visiting position at the Institute for Advanced Study (IAS) in Princeton, on a sabbatical leave from MIT, but he actually lived in New York City. The IAS at the time "was known to be about the dullest place you could find"[1] and Nash used to hang around the Courant Institute, which was close to home and full of activity. That is how he came across a problem that mathematicians had been trying to solve for quite a while. The story goes that Louis Nirenberg, at the time a young professor at Courant, was the person responsible for the unveiling: "(...) it was a problem that I was interested in and tried to solve. I knew lots of people interested in this problem, so I might have suggested it to him, but I'm not absolutely sure", said Nirenberg recently in an interview with the Notices of the AMS (cf. [9]).

As many other paramount questions of 20th-century Mathematics, it all started with one of Hilbert's problems, presented on the occasion of the 1900 International Congress of Mathematicians in Paris, namely the 19th problem: *Are the solutions of regular problems in the calculus of variations always necessarily analytic?* A simple example of such a problem is, in modern terminology, the problem of minimizing a functional

$$\min_{w \in \mathcal{A}} \int_U L(Dw(x))\, dx$$

where $U \subset \mathbb{R}^n$ is a bounded and smooth domain, the Lagrangian $L(\xi)$ is a smooth (possibly nonlinear) scalar function defined on $\mathbb{R}^n$ and $\mathcal{A}$ is a set of admissible functions (typically, the elements of a certain function space satisfying a boundary condition like $w = g$ on ∂U, for a given g). The question is to prove that, given the smoothness of L, the minimizer (assuming it exists) is also smooth.

[1] Cathleen Morawetz, quoted in [7].

Problems of this type are related to elliptic equations in that a minimizer u^* is a weak solution of the associated Euler-Lagrange equation

$$\sum_{j=1}^{n} \left(L_{\xi_j}(Du^*) \right)_{x_j} = 0 \quad \text{in } U.$$

This equation can be differentiated with respect to x_k, to give that, for any $k = 1, 2, \ldots, n$, the partial derivative $(u^*)_{x_k} =: v_k$ satisfies a linear PDE of the form

$$\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i} \right)_{x_j} = 0 \qquad \text{in } U, \tag{3.1}$$

with coefficients $a^{ij}(x) := L_{\xi_i \xi_j}(Du^*(x))$. The PDE is then elliptic provided L is assumed to be convex.

In the 1950s, regularity theory for elliptic equations was essentially based on Schauder's estimates, which, roughly speaking, guarantee that if $a^{ij} \in C^{k,\alpha}$ then the solutions of (3.1) are of class $C^{k+1,\alpha}$, for $k = 0, 1, \ldots$ So if it could be shown that $u^* \in C^{1,\alpha}$ then $a^{ij} = L_{\xi_i \xi_j}(Du^*)$ would belong to $C^{0,\alpha}$, each v_k to $C^{1,\alpha}$ and u^* to $C^{2,\alpha}$; a bootstrap argument would then solve Hilbert's 19th problem.

Meanwhile, the existence theory had been developed through the use of direct methods: the minimization problem has a unique solution provided L, apart from satisfying natural growth conditions like

$$|L(\xi)| \leq C \, |\xi|^2,$$

is also coercive and uniformly convex. The notion of solution had to be conveniently extended and the admissible set $\mathcal{A}$ taken to be the set of functions that, together with their first weak derivatives, belong to L^2, i.e., that belong to the Sobolev space H^1.

So the existence theory gave a minimizer $u^* \in H^1$, and the missing step for the regularity problem to be solved was

$$u^* \in H^1 \Longrightarrow u^* \in C^{1,\alpha},$$

i.e., from first derivatives in L^2 to Hölder continuous first derivatives.

Regarding the elliptic PDE (3.1), the question was to show that weak solutions are Hölder continuous assuming only the measurability and the boundedness of the coefficients (since it is also customary to assume $|D^2 L(\xi)| \leq C$). We could then close the bootstrap argument

$$u^* \in H^1 \Rightarrow L_{\xi_i \xi_j}(Du^*) = \boxed{a^{ij} \in L^\infty \Rightarrow v_k \in C^{0,\alpha}} \Rightarrow u^* \in C^{1,\alpha}$$

The problem was solved by C.B. Morrey in 1938 for the special case $n = 2$, but the techniques he employed were typically two-dimensional, involving complex analysis and quasi-conformal mappings. The n-dimensional problem remained open until the late 50's as the available regularity theory only worked if the coefficients were already somewhat regular (at least continuous), since it was based on perturbation arguments and comparison of the solutions with harmonic functions.

3.2. **De Giorgi-Nash theory.** The problem would not resist the genius of Ennio De Giorgi and John Nash. The two mathematicians worked totally unaware of each other's progress and solved the problem using entirely different methods [1, 7].

Consider equation (3.1), with coefficients only assumed to be measurable and bounded, with

$$\|a^{ij}\|_{L^\infty} \leq \Lambda,$$

and to satisfy the uniform ellipticity condition (1.2).

To simplify the writing, we assume from now on that

$$U = B_1 := \{x \in \mathbb{R}^n : |x| < 1\}\,.$$

Theorem 4. *Every weak solution of* (3.1) *is locally bounded and, for any* $\theta \in (0,1)$*, we have*

$$\sup_{B_\theta} u \leq \frac{C(n,\lambda,\Lambda)}{(1-\theta)^{n/2}}\, \|u^+\|_{L^2(B_1)}.$$

Proof. Let $k \geq 0$ and $\eta \in C_c^\infty(B_1)$, put $v = (u-k)^+$ and take $\varphi = v\eta^2$ as test function in (1.3) with $f \equiv 0$. Using the assumptions and Young's inequality, we get

$$\int_{B_1} |Dv|^2 \eta^2 \, dx \leq \frac{4\Lambda^2}{\lambda^2} \int_{B_1} v^2 |D\eta|^2 \, dx. \tag{3.2}$$

These Caccioppoli inequalities on level sets of u will be the building blocks of the whole theory, and once they are obtained, the PDE can be forgotten: the problem becomes purely analytic.

Next, by Hölder and Sobolev inequalities ($2^* = 2n/(n-2)$ being the Sobolev exponent),

$$\begin{aligned} \int_{B_1} (v\eta)^2 \, dx &\leq \left(\int_{B_1} (v\eta)^{2^*} dx \right)^{\frac{2}{2^*}} |\{v\eta \neq 0\}|^{1-\frac{2}{2^*}} \\ &\leq C(n)\ |\{v\eta \neq 0\}|^{\frac{2}{n}} \int_{B_1} |D(v\eta)|^2 \, dx \end{aligned}$$

and since, due to (3.2),

$$\int_{B_1} |D(v\eta)|^2 \, dx \le \left(\frac{4\Lambda^2}{\lambda^2} + 1 \right) \int_{B_1} v^2 |D\eta|^2 \, dx,$$

we arrive at

$$\int_{B_1} (v\eta)^2 \, dx \le C(n, \lambda, \Lambda) \; |\{ v\eta \neq 0 \}|^{\frac{2}{n}} \int_{B_1} v^2 |D\eta|^2 \, dx.$$

Now for fixed $0 < r < R < 1$, choose the cutoff function $\eta \in C_c^\infty(B_R)$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ in B_r and $|D\eta| \le \frac{2}{R-r}$. Denoting, for $\rho > 0$,

$$A(k, \rho) = \{ \, x \in B_\rho \, : \, u(x) > k \, \} \, ,$$

we obtain (with $C \equiv C(n, \lambda, \Lambda)$)

$$\int_{A(k,r)} (u-k)^2 \, dx \le \frac{C}{(R-r)^2} \; |A(k, R)|^{\frac{2}{n}} \int_{A(k,R)} (u-k)^2 \, dx.$$

For $k > h$ and $0 < \rho < 1$, we have

$$\int_{A(k,\rho)} (u-k)^2 \, dx \le \int_{A(k,\rho)} (u-h)^2 \, dx \le \int_{A(h,\rho)} (u-h)^2 \, dx$$

and

$$\int_{A(h,\rho)} (u-h)^2 \, dx \ge \int_{A(k,\rho)} (u-h)^2 \, dx \ge \int_{A(k,\rho)} (k-h)^2 \, dx = (k-h)^2 |A(k, \rho)|.$$

We can then estimate

$$\begin{aligned} \int_{A(k,r)} (u-k)^2 \, dx \quad &\le \quad \frac{C}{(R-r)^2} \; |A(k, R)|^{\frac{2}{n}} \int_{A(k,R)} (u-k)^2 \, dx \\ &\le \quad \frac{C}{(R-r)^2} \, \frac{1}{(k-h)^{\frac{4}{n}}} \left(\int_{A(h,R)} (u-h)^2 \, dx \right)^{1+\frac{2}{n}} \end{aligned}$$

or, equivalently, with $\psi(s, \rho) = \|(u-s)^+\|_{L^2(B_\rho)}$,

$$\psi(k, r) \le \frac{C}{R-r} \, \frac{1}{(k-h)^{\frac{2}{n}}} \, \psi(h, R)^{1+\frac{2}{n}}, \tag{3.3}$$

for any $k > h > 0$ and $0 < r < R < 1$.

We are now ready for the iteration scheme devised by De Giorgi. Define, for $m = 0, 1, 2, \ldots$,

$$\begin{aligned} k_m \quad &= \quad k \, (1 - \frac{1}{2^m}) \\ r_m \quad &= \quad \frac{1}{2} \, (1 + \frac{1}{2^m}), \end{aligned}$$

where k is to be determined. Observe that $k_0 = 0$ and the sequence $(k_m)_m$ is increasing, with limit $k_\infty = k$, and that $r_0 = 1$ and the sequence $(r_m)_m$ is decreasing, with limit $r_\infty = \frac{1}{2}$.

Due to (3.3), we then have, for $m = 0, 1, 2, \ldots$,

$$\psi(k_m, r_m) \leq C \, \frac{2^{m+1+\frac{2m}{n}}}{k^{\frac{2}{n}}} \, \psi(k_{m-1}, r_{m-1})^{1+\frac{2}{n}} \tag{3.4}$$

and will prove, by induction, that, for some $\gamma > 1$,

$$\psi(k_m, r_m) \leq \frac{\psi(k_0, r_0)}{\gamma^m} \ , \ \forall \, m = 0, 1, 2, \ldots \tag{3.5}$$

if k is chosen sufficiently large. In fact, it is trivial that (3.5) holds for $m = 0$; now suppose it holds for $m - 1$ and write

$$\begin{aligned} \psi(k_{m-1}, r_{m-1})^{1+\frac{2}{n}} &\leq \left\{ \frac{\psi(k_0, r_0)}{\gamma^{m-1}} \right\}^{1+\frac{2}{n}} \\ &= \frac{\psi(k_0, r_0)^{\frac{2}{n}}}{\gamma^{\frac{2m}{n} - (1+\frac{2}{n})}} \, \frac{\psi(k_0, r_0)}{\gamma^m}. \end{aligned}$$

From (3.4), we obtain

$$\psi(k_m, r_m) \leq 2C\gamma^{1+\frac{2}{n}} \, \frac{\psi(k_0, r_0)^{\frac{2}{n}}}{k^{\frac{2}{n}}} \, \frac{2^{m(1+\frac{2}{n})}}{\gamma^{\frac{2m}{n}}} \, \frac{\psi(k_0, r_0)}{\gamma^m}$$

and choose first $\gamma > 1$ such that $\gamma^{\frac{2}{n}} = 2^{1+\frac{2}{n}}$ and then k large enough so that

$$2C\gamma^{1+\frac{2}{n}} \, \frac{\psi(k_0, r_0)^{\frac{2}{n}}}{k^{\frac{2}{n}}} \leq 1 \ \Leftarrow \ k = C \, \psi(k_0, r_0)$$

where $C \equiv C(n, \lambda, \Lambda)$.

Finally let $m \to \infty$ in (3.5) to get $\psi(k, \frac{1}{2}) \leq 0$, i.e.,

$$\|(u - k)^+\|_{L^2(B_{\frac{1}{2}})} = 0.$$

Hence

$$\sup_{B_{\frac{1}{2}}} u \leq C \, \|u^+\|_{L^2(B_1)}. \tag{3.6}$$

A lower bound can be obtained in a similar manner. Redoing the proof with that purpose in mind is an excellent exercise, and the reader is strongly encouraged to try it.

The refinement of the estimate follows a standard dilation argument. □

Before we proceed, we need some preparatory material. We start with the definitions of subsolution and supersolution.

Definition 3. *A function $u \in H^1_{loc}(U)$ is a weak subsolution of* (3.1) *if*

$$\int_U a^{ij} u_{x_i} \varphi_{x_j}\, dx \leq 0, \quad \forall \varphi \in H^1_0(U),\ \varphi \geq 0. \tag{3.7}$$

It is a supersolution if (3.7) *holds with $\leq$ replaced by $\geq$.*

We now prove two lemmas: the first, due to Moser, concerns the composition with convex functions; the second is a Poincaré inequality.

Lemma 4. *Let $\Phi \in C^{0,1}_{loc}(\mathbb{R})$ be convex, with $\Phi' \leq 0$. If u is a supersolution of* (3.1) *then $v = \Phi(u)$ is a subsolution as long as it belongs to $H^1_{loc}(U)$.*

Proof. If $\Phi \in C^2_{loc}(\mathbb{R})$, we have $\Phi'' \geq 0$. Take $\varphi \in H^1_0(U)$ with $\varphi \geq 0$. Then, using the ellipticity,

$$\begin{aligned}
\int_U a^{ij} v_{x_i} \varphi_{x_j}\, dx &= \int_U a^{ij} \Phi'(u) u_{x_i} \varphi_{x_j}\, dx \\
&= -\int_U a^{ij} u_{x_i} (-\Phi'(u)\varphi)_{x_j}\, dx - \int_U (a^{ij} u_{x_i} u_{x_j}) \Phi''(u) \varphi\, dx \\
&\leq 0,
\end{aligned}$$

since $-\Phi'(u)\varphi \in H^1_0(U)$ is non-negative.

For the general case, consider a standard mollification $\Phi_\epsilon(s) = \rho_\epsilon * \Phi(s)$, which remains convex. Then $\Phi'_\epsilon(s) = \rho_\epsilon * \Phi'(s) \leq 0$ and $\Phi''_\epsilon(s) \geq 0$. We then have that $\Phi_\epsilon(u)$ is a subsolution. The result is obtained by passing to the limit in ϵ, using the a.e. convergence $\Phi'_\epsilon(s) \to \Phi'(s)$ and Lebesgue's Dominated Convergence Theorem. □

Lemma 5. *For every $\epsilon > 0$, there exists a constant $C = C(\epsilon, n)$ such that for every $u \in H^1(B_1)$ satisfying*

$$|\{x \in B_1 \,:\, u = 0\}| \geq \epsilon |B_1|,$$

there holds

$$\int_{B_1} u^2\, dx \leq C \int_{B_1} |Du|^2\, dx.$$

Proof. The proof is by contradiction. Supposing the result is false, construct a sequence $(u_m)_m \subset H^1(B_1)$ such that

$$|\{x \in B_1 \,:\, u_m = 0\}| \geq \epsilon |B_1|,$$

$$\int_{B_1} u_m^2\, dx = 1 \qquad \text{and} \qquad \int_{B_1} |Du_m|^2\, dx \longrightarrow 0 \quad \text{as } m \to \infty.$$

We just need to take $C = m$ and normalize the u_m in the L^2-norm. The sequence $(u_m)_m$ is then bounded in the reflexive Sobolev space $H^1(B_1)$ and thus possesses a weakly convergent subsequence. Relabelling if necessary, we have

$$u_m \longrightarrow u_\infty,$$

weakly in $H^1(B_1)$ and strongly in $L^2(B_1)$ (due to Rellich–Kondrachov Theorem), for some $u_\infty \in H^1(B_1)$.

But $u_\infty = \pm\sqrt{1/|B_1|}$; in fact, it is constant since

$$\|Du_\infty\|_2 \leq \liminf \|Du_m\|_2 = 0,$$

and we also have

$$\|u_\infty\|_2 = \lim \|u_m\|_2 = 1.$$

Thus

$$\begin{aligned} 0 &= \lim_{m\to\infty} \int_{B_1} |u_m - u_\infty|^2 \, dx \\ &\geq \lim_{m\to\infty} \int_{B_1\cap\{u_m=0\}} |u_m - u_\infty|^2 \, dx \\ &\geq \frac{\epsilon|B_1|}{|B_1|} \\ &= \epsilon, \end{aligned}$$

a contradiction. □

We next prove a density theorem, stating that if the set where the solution is large occupies a significant portion of B_1, then the solution is indeed large everywhere inside a smaller ball. It converts a statement *in measure* into a statement holding everywhere.

Theorem 5. *Let $u \in H^1(B_2)$ be a non-negative weak solution of* (3.1) *in B_2, for which*

$$\exists\, \epsilon > 0 \;:\; \frac{|B_1 \cap \{u \geq 1\}|}{|B_1|} \geq \epsilon.$$

Then there exists a constant $C = C(\epsilon, n, \lambda, \Lambda) > 0$ such that

$$\inf_{B_{\frac{1}{2}}} u \geq C.$$

Proof. We will prove the theorem for $u+\delta \geq \delta > 0$ and then let $\delta \to 0^+$. It is crucial to observe that every $u+\delta$ solves the same PDE and satisfies the assumption of the theorem for the same ϵ since

$$B_1 \cap \{u+\delta \geq 1\} \supseteq B_1 \cap \{u \geq 1\}, \quad \forall \delta > 0.$$

Thus, the constant C appearing in the conclusion of the theorem is independent of δ since it only depends on ϵ, the dimension, and the coefficients.

By Lemma 4,

$$v = \Phi(u) := (\log u)^- \geq 0$$

is a subsolution of (3.1), bounded above by $-\log \delta$. Since it is obvious that the local boundedness holds for subsolutions, (3.6) gives

$$\sup_{B_{\frac{1}{2}}} v \leq C \, \|v\|_{L^2(B_1)}.$$

Now, clearly

$$|B_1 \cap \{v = 0\}| = |B_1 \cap \{u \geq 1\}| \geq \epsilon |B_1|,$$

and Lemma 5 yields

$$\sup_{B_{\frac{1}{2}}} v \leq C \, \|Dv\|_{L^2(B_1)}. \tag{3.8}$$

To bound the right-hand side, test the equation against $\varphi = \xi^2/u$, where $\xi \in C_0^1(B_2)$ and $\xi \equiv 1$ in B_1. We have

$$\begin{aligned} 0 &= \int_{B_2} a^{ij} u_{x_i} \left(\frac{\xi^2}{u}\right)_{x_j} dx \\ &= -\int_{B_2} \frac{a^{ij} u_{x_i} u_{x_j}}{u^2} \xi^2 \, dx + 2\int_{B_2} \frac{a^{ij} u_{x_i} \xi_{x_j}}{u} \xi \, dx \end{aligned}$$

and, using the ellipticity and Cauchy's inequality (1.4), with $\epsilon = \frac{\lambda}{4\Lambda}$, we obtain

$$\begin{aligned} \lambda \int_{B_2} \left|\frac{Du}{u}\right|^2 \xi^2 \, dx &\leq 2\Lambda \int_{B_2} \left|\frac{Du}{u}\right| |D\xi| \xi \, dx \\ &\leq 2\Lambda \left\{ \frac{\lambda}{4\Lambda} \int_{B_2} \left|\frac{Du}{u}\right|^2 \xi^2 \, dx + \frac{\Lambda}{\lambda} \int_{B_2} |D\xi|^2 \, dx \right\} \end{aligned}$$

which yields

$$\int_{B_1} |D \log u|^2 \, dx \leq \int_{B_2} |D \log u|^2 \xi^2 \, dx \leq \frac{4\Lambda^2}{\lambda^2} \int_{B_2} |D\xi|^2 \, dx = C.$$

From (3.8), we get

$$\sup_{B_{\frac{1}{2}}} (\log u)^- = \sup_{B_{\frac{1}{2}}} v \leq C$$

and then

$$\inf_{B_{\frac{1}{2}}} u \geq e^{-C} > 0.$$

$\square$

The basic general idea to obtain results concerning the continuity of a solution of a PDE at a point consists in estimating its oscillation in a nested sequence of concentric balls, centered at the point, and showing that it converges to zero as the balls shrink to the point. If this can be measured quantitatively, it gives a modulus of continuity.

Denote the oscillation of a function u in B_r with

$$\operatorname{osc}(u,r) := \sup_{B_r} u - \inf_{B_r} u.$$

Theorem 6. *Let $u \in H^1(B_2)$ be a weak solution of* (3.1)*. There exists a constant $\gamma = \gamma(n,\lambda,\Lambda) \in (1/2,1)$ such that*

$$\operatorname{osc}(u,1/2) \leq \gamma \operatorname{osc}(u,1).$$

Proof. We already know the solution is locally bounded. Set

$$M = \sup_{B_1} u \qquad \text{and} \qquad m = \inf_{B_1} u.$$

Observe that both $\dfrac{2(u-m)}{M-m}$ and $\dfrac{2(M-u)}{M-m}$ are non-negative solutions of the equation and that

$$\frac{2(u-m)}{M-m} \leq 1 \iff \frac{2(M-u)}{M-m} \geq 1.$$

We then have an alternative: either

$$\left| \left\{ x \in B_1 \,:\, \frac{2(u-m)}{M-m} \geq 1 \right\} \right| \geq \frac{1}{2}\,|B_1|$$

or

$$\left| \left\{ x \in B_1 \,:\, \frac{2(M-u)}{M-m} \geq 1 \right\} \right| \geq \frac{1}{2}\,|B_1|\,.$$

In the first alternative, apply Theorem 5 to $\dfrac{2(u-m)}{M-m} \geq 0$ to obtain

$$\inf_{B_{\frac{1}{2}}} \frac{2(u-m)}{M-m} \geq C \iff \inf_{B_{\frac{1}{2}}} u \geq m + \frac{C}{2}(M-m),$$

for some $0 < C < 1$.

In the second alternative, apply Theorem 5 to $\dfrac{2(M-u)}{M-m} \geq 0$ to obtain

$$\inf_{B_{\frac{1}{2}}} \frac{2(M-u)}{M-m} \geq C \iff \sup_{B_{\frac{1}{2}}} u \leq M - \frac{C}{2}(M-m),$$

for the same $C > 0$. In either case, we obtain

$$\begin{aligned} \operatorname{osc}(u, 1/2) &= \sup_{B_{\frac{1}{2}}} u - \inf_{B_{\frac{1}{2}}} u \\ &\leq M - m - \frac{C}{2}(M-m) \\ &= \left(1 - \frac{C}{2}\right)(M-m) \\ &= \gamma \operatorname{osc}(u, 1), \end{aligned}$$

where $\gamma := 1 - C/2 \in (1/2, 1)$ because $C \in (0, 1)$. □

Theorem 7. *Let $u \in H^1(B_2)$ be a weak solution of* (3.1). *There exist constants $C > 0$ and $\alpha \in (0, 1)$ such that, for every $0 < r < R < 1$,*

$$\operatorname{osc}(u, r) \leq C \left(\frac{r}{R}\right)^\alpha \operatorname{osc}(u, R).$$

Proof. Take $0 < R < 1$. The reduction of the oscillation of Theorem 6 gives

$$\operatorname{osc}(u, R/2) \leq \gamma \operatorname{osc}(u, R)$$

and, by induction, we trivially get

$$\operatorname{osc}(u, 2^{-k} R) \leq \gamma^k \operatorname{osc}(u, R); \quad k = 1, 2, \ldots$$

Now, take $0 < r < R$, and choose k such that $2^{-(k+1)} R \leq r < 2^{-k} R$. We have

$$\begin{aligned} \operatorname{osc}(u, r) &\leq \operatorname{osc}(u, 2^{-k} R) \\ &\leq \gamma^k \operatorname{osc}(u, R) \\ &\leq \gamma^{-1} \left(\frac{r}{R}\right)^\alpha \operatorname{osc}(u, R), \end{aligned}$$

for $\alpha = -\frac{\log \gamma}{\log 2}$. Note that $\alpha \in (0, 1)$ (because $\gamma \in (1/2, 1)$) and that

$$\gamma^{k+1} = \left(2^{-\alpha}\right)^{k+1} = \left(2^{-(k+1)}\right)^\alpha \leq \left(\frac{r}{R}\right)^\alpha.$$

The result follows with $C = \gamma^{-1}$. □

We finally obtain the celebrated result of De Giorgi and Nash.

Theorem 8. *Every bounded weak solution of* (3.1) *is locally Hölder continuous, i.e., for every compact $K \subset U$, there exist constants $C > 0$ and $\alpha \in (0,1)$, depending only on U, K, λ, Λ, such that*

$$|u(x) - u(y)| \leq C\|u\|_\infty |x-y|^\alpha, \quad \forall\, x, y \in K.$$

Proof. Assume again that $U = B_1$. Take $K \subset B_1$ compact and let

$$R = \frac{1}{2}\mathrm{dist}(K, \partial B_1).$$

Fix $x, y \in K$ and put $r = |x-y|$. If $r < R$ then, applying Theorem 7 in the balls $B_r(x) \subset B_R(x) \subset B_1$, we get

$$\begin{aligned} |u(x) - u(y)| &\leq \operatorname{osc}(u, r) \\ &\leq C\left(\frac{r}{R}\right)^\alpha \operatorname{osc}(u, R) \\ &= \frac{2^\alpha C\|u\|_\infty}{[\mathrm{dist}(K, \partial B_1)]^\alpha}|x-y|^\alpha \\ &= C\|u\|_\infty |x-y|^\alpha. \end{aligned}$$

If $r > R$ then

$$\begin{aligned} |u(x) - u(y)| &\leq 2\|u\|_\infty \\ &\leq 2\|u\|_\infty \left(\frac{r}{R}\right)^\alpha \\ &= \frac{2^{\alpha+1}\|u\|_\infty}{[\mathrm{dist}(K, \partial B_1)]^\alpha}|x-y|^\alpha \\ &= C\|u\|_\infty |x-y|^\alpha. \end{aligned}$$

□

3.3. **Moser's Harnack inequality.** J. Moser made significant contributions to the theory. He first gave in [5] a new proof of De Giorgi's theorem, using the simple general principle that the estimates (3.2) hold for any convex function $f(u)$ of a solution u; the results were obtained by applying such estimates to powers $f(u) = |u|^p$, $p \geq 1$, and to the logarithmic function $(\log u)^-$ (cf. Theorem 5). Then he proved the following Harnack inequality for elliptic equations (cf. [6]).

Theorem 9. *If u is a non-negative weak solution of* (3.1) *and K is a compact subset of U, then*

$$\max_K u \leq C \min_K u,$$

where $C \equiv C(U, K, \lambda, \Lambda)$.

The proof of the Harnack inequality made no use of the Hölder continuity of the solutions, which in turn is a simple consequence of that fact, as Moser showed in the paper.

Theorem 10. *Harnack implies Hölder.*

Proof. Assume again that $U = B_1$. Let, for $0 < r < 1$,

$$M(r) = \max_{\overline{B_r}} u \ , \qquad m(r) = \min_{\overline{B_r}} u$$

and apply Harnack's inequality to the domains B_r and $\overline{B_{\frac{r}{2}}}$ to get

$$\begin{aligned} M(r) - m(r/2) &= \max_{\overline{B_{\frac{r}{2}}}} (M(r) - u) \\ &\leq C \min_{\overline{B_{\frac{r}{2}}}} (M(r) - u) \\ &= C \ (M(r) - M(r/2)) \end{aligned}$$

since $M(r) - u$ is a non-negative solution in B_r. The same is true for $u - m(r)$, and we obtain similarly

$$M(r/2) - m(r) \leq C \left(m(r/2) - m(r) \right).$$

Adding these two inequalities, we get

$$M(r/2) - m(r/2) \leq \frac{C-1}{C+1} \ (M(r) - m(r))$$

or, with $\gamma = \frac{C-1}{C+1} < 1$,

$$\operatorname{osc}(u, r/2) \leq \gamma \operatorname{osc}(u, r).$$

This reduction of the oscillation now gives the Hölder continuity as before. $\square$

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