PhD Program in Mathematics UC|UP

Partial Differential Equations

Spring 2008

HOMEWORK 1

1. Let u be the real function defined in the open set $U = \{(x,y) \in \mathbb{R}^2 \ : \ y > |x|\}$ by
$$u(x,y) = y^2 - x^2.$$
   (a) Show that u is harmonic in U and vanishes on ∂U.
   (b) Show that u assumes in U arbitrarily large values.
   (c) Explain why this is not a counter-example to the maximum principle.

2. Given $U \subset \mathbb{R}^n$ open, let $u \in H^1(U)$ be a **bounded** weak solution of the elliptic PDE
$$-\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i}\right)_{x_j} = 0 \quad \text{in } U.$$
Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and **convex**, and set $w = \phi(u)$. Show that
$$\int_U \sum_{i,j=1}^{n} a^{ij} w_{x_i} v_{x_j} \, dx \leq 0, \quad \forall\, v \in H_0^1(U),\ v \geq 0.$$
This means that w is a *weak subsolution* of the equation.

3. Let $f \in L^2(\mathbb{R}^n)$ and $c : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $c(0) = 0$ and $c' \geq 0$. Assume $u \in H^1(\mathbb{R}^n)$ has compact support and is a weak solution of the semilinear PDE
$$-\Delta u + c(u) = f \quad \text{in } \mathbb{R}^n.$$
Use the difference quotient method of Nirenberg to show that $u \in H^2(\mathbb{R}^n)$.

Due on February 26, 2008