There is no mni clutter whose core is the projective plane of order 3

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Índice

Clutter definition

Definition 1

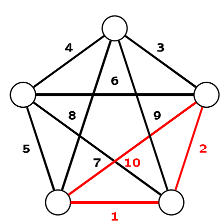
A clutter is a pair $\mathcal{C} \equiv (V, E)$, where E is a family of subsets of a finite and nonempty set V such that no element in E contains another. V is the *ground set*; E as the *edge set*.

It is helpful to describe clutters by their incidence matrices

$\mathcal{C} \equiv \mathcal{T}(K_5)$

$$E \quad \begin{array}{r} \\ 12(10) \\ 189 \\ 157 \\ 38(10) \\ 56(10) \\ 458 \\ 239 \\ 267 \\ 479 \\ 346 \end{array} \overbrace{\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}}^{V} $$

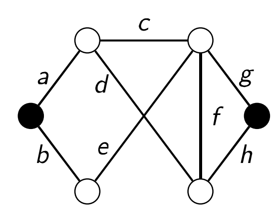
$$\begin{array}{r|cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 12(10) & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 189 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 157 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 38(10) & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 56(10) & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 458 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 239 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 267 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 479 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 346 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}$$



Clutter, examples

the clutter of the (s,t)-paths in a given graph;
the clutter of the minimal edge (s,t)-cutsets in a given graph:

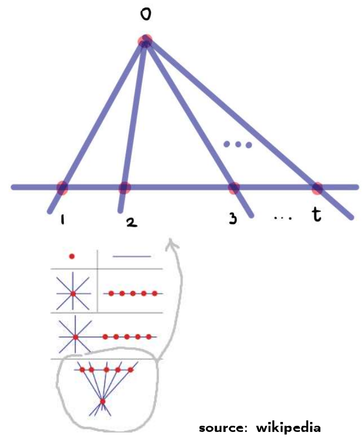
$$\begin{array}{r} \\ acfh \\ acg \\ adfg \\ adh \\ befh \\ beg \\ becdh \end{array}\begin{array}{cccccccc} a & b & c & d & e & f & g & h \\ \hline 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}, \qquad \begin{array}{cccccccc} a & b & c & d & e & f & g & h \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{array}\begin{array}{l} \\ ab \\ ae \\ gh \\ bcd \\ cde \\ dfg \\ acfg \\ bcfh \\ cef \end{array}$$



Clutter, examples

the *Delta* Δ_t, for some $t = 2, 3, \ldots$

$$
\begin{array}{r|c|cccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1234567 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
01 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
02 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
03 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
04 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
05 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
06 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$



source: wikipedia

blocker, cover polyhedron

Definition 2

The *blocker* of a clutter $\mathcal{C} \equiv (V, E)$ is a clutter $\text{bl}\,\mathcal{C} \equiv (V, F)$ whose edges are the *minimal covers (transversals)* of $\mathcal{C}$, *i.e.*,

$$B \in F \qquad \text{iff} \qquad B \cap A \neq \emptyset, \forall A \in E.$$

$\mathcal{C} = \text{bl}\,\text{bl}\,\mathcal{C}$, and if $A = M(\mathcal{C})$ and $B = M(\text{bl}\,\mathcal{C})$ then

$$\text{ext}\, Q(A) \cap \mathbb{Z}^n = \{\text{rows of } B\} \equiv F,$$
$$\text{ext}\, Q(B) \cap \mathbb{Z}^n = \{\text{rows of } A\} \equiv E,$$

where $Q(A)$ is the *cover polyhedron*

$$Q(A) = \{x \colon Ax \geq 1,\ x \geq 0\}.$$

blocker, examples

- the blocker of $\mathcal{T}(K_5)$ is the clutter composed of 10 triangles plus one nonadjacent edge, 12 hamiltonian cycles, and the 5 K_4's inside the K_5- 27 edges total.
- if $\mathcal{C}$ is just the edge set of a simple graph G, $\text{bl}\,\mathcal{C}$ are the minimal (edge) covers of G.
- the blocker of the clutter of (s,t)-paths is the clutter of minimal edge (s,t)-cutsets,
- the blocker of the clutter of trees is the clutter of minimal edge cutsets (of a connected graph)
- the blocker of Δ_t is again Δ_t (selfblocking)
- (...)

Minors: deletion and contraction

Given a clutter $\mathcal{C} \equiv (V, E)$, a *minor* $\mathcal{H} \equiv \mathcal{C} \setminus I / J$ is a clutter that results from $\mathcal{C}$ after sequentially applying each one of the following (where $I, J \subseteq V, I \cap J = \emptyset$)

Definition 3: *deletion of $\mathcal{C}$ by j*

the clutter $\mathcal{C} \setminus j$ has vertex set $V \setminus \{j\}$ and edge set the inclusion-minimal sets in E that do not include j.

Definition 4: *contraction of $\mathcal{C}$ by j*

the clutter $\mathcal{C} / j$ has vertex set $V \setminus \{j\}$ and edge set the inclusion-minimal sets in $A \setminus \{j\}$, for every $A \in E$.

$$
\begin{array}{r|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
135 & 1 & 0 & 1 & 0 & 1 \\
134 & 1 & 0 & 1 & 1 & 0 \\
124 & 1 & 1 & 1 & 0 & 0 \\
235 & 0 & 1 & 1 & 0 & 1 \\
24 & 0 & 1 & 0 & 1 & 0 \\
345 & 0 & 0 & 1 & 1 & 1
\end{array}
\equiv M(\mathcal{C} \setminus 1)
\qquad
\begin{array}{r|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
135 & 1 & 0 & 1 & 0 & 1 \\
134 & 1 & 0 & 1 & 1 & 0 \\
124 & 1 & 1 & 1 & 0 & 0 \\
235 & 0 & 1 & 1 & 0 & 1 \\
24 & 0 & 1 & 0 & 1 & 0 \\
345 & 0 & 0 & 1 & 1 & 1
\end{array}
\equiv M(\mathcal{C} / 1)
$$

$$\mathcal{C} \setminus i / j \equiv \mathcal{C} / j \setminus i, \qquad \text{bl}(\mathcal{C} \setminus I / J) \equiv \text{bl}(\mathcal{C}) / I \setminus J.$$

ideal/minimally non-ideal

Definition 5

A clutter $\mathcal{C}$ is *ideal* if, for $A \equiv M(\mathcal{C})$, the cover polyhedron $Q(A)$ is integral, *i.e.*, all its extreme points are defined by integer components.

$\mathcal{C}$ is ideal if and only if $\mathrm{bl}\,\mathcal{C}$ is ideal

Definition 6

A clutter $\mathcal{C}$ is *minimally nonideal*, mni for short, if $\mathcal{C}$ is not ideal but all its proper minors are.

$\mathcal{C}$ is mni if and only if $\mathrm{bl}\,\mathcal{C}$ is mni

Examples

$$\text{ideal}\ \left\{ \underbrace{\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 124 \\ 135 \\ 236 \\ 456 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}}_{M(Q_6)}, \right. \quad \underbrace{\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 13 \\ 124 \\ 235 \\ 346 \\ 45 \\ 156 \\ 26 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}}_{M(\mathrm{bl}\,Q_6)}$$

$$\text{mni}\ \left\{ \underbrace{\begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 124 \\ 235 \\ 346 \\ 457 \\ 561 \\ 672 \\ 713 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}}_{M(\mathcal{F}_7)}, \right. \quad \underbrace{\begin{array}{cc} & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 123456 \\ 01 \\ 02 \\ 03 \\ 04 \\ 05 \\ 06 \end{array} & \left[\begin{array}{c:cccccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hdashline 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}}_{M(\Delta_6)}$$

Lehman matrix

Definition 7

Matrix Y, $n \times n$ of zeros and ones, is a *Lehman matrix* if there exists Z, $n \times n$ of zeros and ones, and a positive integer d such that $YZ^T = dI + \mathbb{1}\mathbb{1}^T$.

For example,

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}}_{Y} \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}^T}_{Z^T} = \underbrace{\begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}}_{YZ^T \equiv (\mathbf{1})I + \mathbb{1}\mathbb{1}^T}$$

Theorem 1: (Bridges and Ryser,1969)

If Y is a Lehman matrix then there is a pair of factors (r, s) of $n+d$ such that Y é r-regular and Z is s-regular. Besides, $YZ^T = Z^TY$.

Lehman's Theorem

Theorem 2: (Lehman,1990)

Let $\mathcal{C} \neq \Delta_t$ be a clutter with $n \geq 3$ vertices. The clutter $\mathcal{C}$ is mni if and only if, for $A = M(\mathcal{C})$ and $B = M(\mathrm{bl}\,\mathcal{C})$:

1. $Q(A)$ (resp. $Q(B)$) has a unique extreme point $\mathring{x}$ (resp. $\mathring{y}$) with some fractional component;
2. $0 < \mathring{x}$ (resp. $\mathring{y}$) $< \mathbb{1}$ and $\mathring{x}$ (resp. $\mathring{y}$) is defined by a unique core $\hat{A}$ (resp. $\hat{B}$).
3. the pair $(\hat{A}, \hat{B})$ is Lehman, where $\hat{A}$ is r-regular and $\hat{B}$ is s-regular, for two positive integers r, s such that, for some positive integer d, $rs = n + d$.
4. every row of A (resp. B) that is not a row of $\hat{A}$ (resp. $\hat{B}$) has more than r (resp. s) ones.

Thus, if $\mathcal{C}$ is mni, the rows of $A \equiv M(\mathcal{C})$ can be permuted so that the first $n \equiv V(\mathcal{C})$ define a Lehman matrix (r-regular), and the remaining rows have at least $r+1$ ones.

The clutter core($\mathcal{C}$) is such that $\hat{A} \equiv M(\mathrm{core}(\mathcal{C}))$. The fact that $\hat{A}$ is a Lehman matrix will naturally leads us to investigate the link with projective planes.

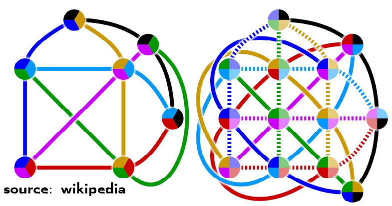
Projective Planes: definition, examples

Definition 8: (*e.g.*, Gossett'2009)

A *projective plane (finite and nondegenerate)* $\mathcal{C}$ is an ordered pair (V, E), where V denotes a finite nonempty set (the points) and E denotes a family of subsets of V (the lines) - satisfying the following three axioms:

- **(AX1)** Any two distinct points are in one and only one common line;
- **(AX2)** Any two distinct lines contain one and only one common point;
- **(AX3)** There exist four points, no three of which are in a common line.

V is associated with points on the plane,
E is associated with lines on the plane that pass through some of the points in V



source: wikipedia

Projective Planes: properties

- If $\mathcal{C} \equiv (V, E)$ is a projective plane then there is some positive integer $n \geq 2$ such that

$$|V| = |E| = n^2 + n + 1 \equiv n(n+1) + 1,$$

and the (point-line) incidence matrix A of $\mathcal{C}$ is $(n+1)$-regular. Moreover, A is a Lehman matrix because

$$A^T A = AA^T = nI + ee^T, \qquad \det(A) = \pm(n+1)n^{n(n+1)/2}.$$

- Then, we say that $\mathcal{C}$ is is projective plane of order n. There are unique projective planes of order 2 and 3 (Fano plane, or $PG(2,2)$, and $PG(2,3)$). Others?

$$\overset{1}{2}, \overset{1}{3}, \overset{1}{4}, \overset{1}{5}, \overset{0}{6}, \overset{1}{7}, \overset{1}{8}, \overset{4}{9}, \overset{0}{10}, 11, 12, 13, \overset{0}{14}, 15, \overset{\geq 22}{16},$$

$$17, 18, 19, 20, \overset{0}{21}, \overset{0}{22}, 23, 24, \overset{\geq 193}{25}, 26, \overset{\geq 13}{27}, 28, 29, \overset{0}{30}, , 31, 32, \ldots$$

- Every projective plane $\mathcal{C}$ defines a clutter $\mathcal{C}$ such that $A \equiv M(\mathcal{C})$ is Lehman.

Projective Planes and Minimally Non Ideal Clutters

Theorem 3: (Novick'1990)

If $\mathcal{C}$ is a projective plane of order $n \geq 3$ then $\mathcal{C}$ is not mni.

Is it possible to define a mni clutter $\mathcal{C}$ by simply adding a sufficient large number of edges to $\mathcal{C}'$ in such a way that core$(\mathcal{C}) = \mathcal{C}'$?

A blocking set of a projective plane $\mathcal{C}'$ is a cover of the $\mathcal{C}'$ (as clutter). A nontrivial blocking set is a cover of the $\mathcal{C}'$ that does not contain an element (edge) of $\mathcal{C}'$.

Theorem 4: (Mayhew, Pivotto and Royle'2019)

Let $\mathcal{C}$ be a clutter mni such that $\mathcal{C}' \equiv$ core$(\mathcal{C})$ is a projective plane. Then: **(i)** $\mathcal{C}' =$ core(bl $\mathcal{C}$); **(ii)** every edge of $\mathcal{C}$ that is not an edge of $\mathcal{C}'$ is a nontrivial blocking set of $\mathcal{C}'$.

blocker of $PG(2,3)$

The incidence matrix of the $PG(2,3)$ is

$$\begin{array}{c} \\ L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \\ L_6 \\ L_7 \\ L_8 \\ L_9 \\ L_{10} \\ L_{11} \\ L_{12} \\ L_{13} \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc} x & a_1 & a_2 & a_3 & y_1 & y_2 & y_3 & b_1 & b_2 & b_3 & y_4 & u & v \end{array} \\
\left[\begin{array}{c:ccc:ccc:cccc:cc}
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hdashline
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
\hdashline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\hdashline
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}.$$

Theorem 5: Tavares and Soares'2025

The blocker of the $PG(2,3)$ are the edges of the $PG(2,3)$ and the 0-corners of the $PG(2,3)$.

0-corner? Let L_x, L_y, L_z three non conpontual lines of the $PG(2,3)$, such that $x = L_y \cap L_z, y = L_x \cap L_z$, $z = L_x \cap L_y$, are the *corners*. Then, $(L_x \cup L_y \cup L_z) \setminus \{x, y, z\}$ is a 0-corner.

the main result

Theorem 6: Tavares and Soares'2025

If $\mathcal{C}$ is a mni clutter such that core($\mathcal{C}$) is a projective plane of order 3 then $\mathcal{C}$ must be the blocker of the $PG(2,3)$.

But, the blocker of $PG(2,3)$ is not mni because $PG(2,3)$ is not mni. The desired result follows.

Comment: it is not obvious how to check whether the cover polyhedron of the blocker of the $PG(2,3)$ that lies in $\mathbb{R}^{13}$ and it is defined by 247 constraints plus non negativity constraints has more than one fractional extreme point. Maybe with a supercomputer ...

In any case, the theorem above also lets you know what is the second fractional extreme point of the cover polyhedron.

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