Risk and Decision Making (Part II - Game Theory) Exam
MIT/Portugal 2 pages
Professor João Soares January 9th, 2008

---

1. Consider the following non-cooperative bimatrix game, where Player 1 chooses one of the three rows and Player 2 chooses one of the three columns:

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \left[\begin{array}{ccc} (-1,1) & (0,2) & (0,2) \\ (2,1) & (1,-1) & (0,0) \\ (0,0) & (1,1) & (1,2) \end{array} \right] \end{array}.$$

   (a) Find the safety levels and the maxmin strategies for both players.

   (b) Find as many strategic equilibria as you can, including the mixed one.

Solution:

(a) The game is defined by the following matrices,

$$A = \left[\begin{array}{rrr} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{array} \right], \quad B = \left[\begin{array}{rrr} 1 & 2 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{array} \right].$$

In the A matrix, the top row and middle column are dominated. The resulting matrix is

$$\left[\begin{array}{cc} 2 & 0 \\ 0 & 1 \end{array} \right].$$

Player I's safety level is $2/3$, and his maxmin strategy is $p = (0, 1/3, 2/3)$. Now, to compute Player II's safety level, we must consider the matrix

$$B^T = \left[\begin{array}{rrr} 1 & 1 & 0 \\ 2 & -1 & 1 \\ 2 & 0 & 2 \end{array} \right].$$

In the B^T matrix, the first column and second row are dominated. The resulting matrix is

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right].$$

Player II's safety level is $2/3$, and her maxmin strategy is $q = (2/3, 0, 1/3)$.

(b) The top row is strictly dominated, and then the middle column is strictly dominated. Removing them does not lose any equilibria (and leads to the Battle of the Sexes). There are two PSE's, one at (second row,first column), and the other at (third row, third column). There is therefore a third SE given by the equalizing strategies in the Battle of the Sexes, namely, (p, q), where $p = (0, 2/3, 1/3)$ and $q = (1/3, 0, 2/3)$.

■

2. Suppose in the Cournot duopoly model that the two firms have different production costs and different set-up costs. Suppose Player I's cost of producing x is $x+2$, and II's cost of producing y is $3y+1$. Suppose also that the price function is $P(x,y) = 17 - x - y$, where x and y are the amounts produced by I and II respectively. What is the equilibrium production, and what are the players' equilibrium profits?

   Solution: The player's profits are

   $$u_1(x,y) = x(17 - x - y) - (x+2), \qquad u_2(x,y) = y(17 - x - y) - (3y+1).$$

   To find the equilibrium production, we set the partial derivatives to zero:

   $$\begin{aligned} \frac{\partial u_1}{\partial x} &= 16 - 2x - y = 0 \\ \frac{\partial u_1}{\partial y} &= 14 - x - 2y = 0 \end{aligned}$$

   which gives $(x,y) = (6,4)$ as the equilibrium production. The equilibrium profits are

   $$(u_1(6,4), u_2(6,4)) = (34, 15).$$

   ■

3. Consider the cooperative TU bimatrix game:

   $$\begin{bmatrix} (1,5) & (2,2) & (0,1) \\ (4,2) & (1,0) & (2,1) \\ (5,0) & (2,3) & (0,0) \end{bmatrix}.$$

   (a) Find the TU-values.

   (b) Find the associated side payment.

   (c) Find the optimal threat strategies.

   Solution:

   (a) The maximum total payoff is $\sigma = 6$, with payoff (1, 5)[1] The difference matrix is

   $$\begin{bmatrix} -4 & 0 & -1 \\ 2 & 1 & 1 \\ 5 & -1 & 0 \end{bmatrix}.$$

   which has a saddle point in the second row and third column. Hence, $\delta = 1$ so that the TU solution is

   $$\varphi = \left(\frac{\sigma+\delta}{2}, \frac{\sigma-\delta}{2}\right) = \left(\frac{7}{2}, \frac{5}{2}\right).$$

   (b) Once the game is played and the outcome is $(1,5)$, Player II should pay $5/2$ to Player I.

   (c) The threat strategies are $p = (0,1,0)$ and $q = (0,0,1)$.

   ■

[1] or $(4,2)$, alternatively.