Risk and Decision Making (Part II - Game Theory) Homework II
MIT/Portugal 2 pages
Professor João Soares October 16th, 2007

Jed, Ned and Ted are antique dealers who conduct their businesses in separate but adjoining rooms of a common premises. Jed's advertised hours are from 12:00 noon until 4:00 p.m., Ned's hours are from 9:00 a.m. until 3:00 p.m. and Ted's are from 1:00 p.m. until 5:00 p.m. Because the dealers have other jobs and the store is never so busy that one individual could not take of everyone's customers, it is in the dealers' interests to pool their time in minding the store; there is no need for two people between noon and 1:00 p.m. or between 3:00 p.m. and 4:00 p.m., or for three people between 1:00 p.m. and 3:00 p.m. Thus, Jed can arrive later than noon or leave earlier than 4:00 p.m., Ted can arrive later that 1:00 p.m. and Ned can leave earlier than 3:00 p.m. But how much later or earlier? What is a fair allocation of store-minding duty for each of the dealers?

This is exercise 10 from Chapter 4 of *An Introduction to Game-Theoretic Modelling*.

Hand in an individual hand written report and be prepared to fully justify it.

You may discuss the solution process with your colleagues but the actual solving should be truly yours.

Solution: This is a 3-person cooperative game. The schedule without cooperation is given below:

The total number of working hours is 14 but only 8 are necessary. If they cooperate, they will share the benefit of 6 hours. It is convenient to express each coalition benefit as a fraction of the grand coalition benefit as follows:

$$v(\emptyset) = 0 \quad \begin{array}{l} v(\{J\}) = 0 \\ v(\{N\}) = 0 \\ v(\{T\}) = 0 \end{array} \quad \begin{array}{l} v(\{J,N\}) = 1/2 \\ v(\{J,T\}) = 1/2 \\ v(\{N,T\}) = 1/3 \end{array} \quad v(\{J,N,T\}) = 1$$

Now, we should determine a vector $x = (x_J, x_N, x_T)$ that stipulates how the benefits of cooperation will be shared among the players. The vector x must be an imputation

$$x_J + x_N + x_T = 1, \quad x_J, x_N, x_T \geq 0. \tag{1}$$

Furthermore, x must be in the reasonable set, *i.e.*,

$$
\begin{aligned}
x_J &\leq \max\{v(\{J,N,T\})-v(\{N,T\}), v(\{J,N\})-v(\{N\}), v(\{J,T\})-v(\{T\}), v(\{J\})-v(\emptyset)\} \\
&\leq \max\{2/3, 1/2, 1/2, 0\} = 2/3 \qquad (2) \\
x_N &\leq \max\{v(\{J,N,T\})-v(\{J,T\}), v(\{J,N\})-v(\{J\}), v(\{N,T\})-v(\{T\}), v(\{N\})-v(\emptyset)\} \\
&\leq \max\{1/2, 1/2, 1/3, 0\} = 1/2 \qquad (3) \\
x_T &\leq \max\{v(\{J,N,T\})-v(\{J,N\}), v(\{N,T\})-v(\{N\}), v(\{J,T\})-v(\{J\}), v(\{T\})-v(\emptyset)\} \\
&\leq \max\{1/2, 1/3, 1/2, 0\} = 1/2. \qquad (4)
\end{aligned}
$$

Let X denote the reasonable set, *i.e.*, the set of x's that satisfy (1)-(4). The least rational core, denoted $C^+(\epsilon_1)$ is the nonempty set of those x's in the reasonable set such that the excess $e(S,x)$, for each nontrivial subset S of $\{J,N,T\}$, is less than or equal to ϵ_1 and such that ϵ_1 is minimum. Formally, we have

$$\epsilon_1 = \min_{\epsilon,x}\{\epsilon\colon x\in X, e(S,x)\leq\epsilon, \text{ for every nonempty } S\subset\{J,N,T\}\}, \qquad (5)$$

where

$$
\begin{aligned}
e(\{J\},x) &= 0-x_J, \\
e(\{N\},x) &= 0-x_N, \\
e(\{T\},x) &= 0-x_T, \\
e(\{J,N\},x) &= \tfrac{1}{2}-x_J-x_N, \\
e(\{J,T\},x) &= \tfrac{1}{2}-x_J-x_T, \\
e(\{N,T\},x) &= \tfrac{1}{3}-x_N-x_T.
\end{aligned}
$$

Using the tool `solver` of `Excel` we find $\epsilon_1 \equiv 0,2222\ldots$. The x-part of the optimal solution is defined by

$$x_1^* = 0.444\ldots, \qquad x_2^* = 0.277\ldots \qquad x_3^* = 0.277\ldots. \qquad (6)$$

Now, for (6) to define the nucleolus we must have x^* as the optimal solution in every

$$\min_x\left\{e(S,x)\colon x\in X^1\right\}, \qquad (7)$$

accross every nonempty subset S of $\{J,N,T\}$, where $X^1\equiv C^+(\epsilon_1)$. The set X^1 is defined by the constraints (1)-(4) and

$$e(S,x)\leq\epsilon_1, \text{ for every nonempty } S\subset\{J,N,T\}.$$

One can verify using the tool `solver` of `Excel` that x^* is still optimal in every program (7). Hence, x^* defined by (6) is the nucleolus. Thus,

> Jed is entitled to $0.444\ldots\times 6$ hrs $=$ 160 mins and so he may start his shift at 13h20 (instead of noon) and end at 14h40 (instead of 4pm);
>
> Ned is entitled to $0.277\ldots\times 6$ hrs $=$ 100 mins and so he may end his shift at 13h20 (instead of 3pm);
>
> Ted is entitled to $0.277\ldots\times 6$ hrs $=$ 100 mins and so he may start his shift at 14h40 (instead of 1pm).

Other schedules are, of course, possible because there is no constraint on the contiguity of the individual schedules.

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