Risk and Decision Making (Part II - Game Theory)	Mock Exam
MIT/Portugal	5 pages
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1. Consider the game defined by the Kuhn tree of Figure 1.

   (a) Describe the game in plain english and find its equivalent strategic form.

   (b) Should each Player use their maxmin (mixed) strategy, what is the expected outcome of the game?

   (c) Find all pure strategic equilibria of this game.

   (d) Should Player I allow Player II to make its move what is the natural outcome of this game?

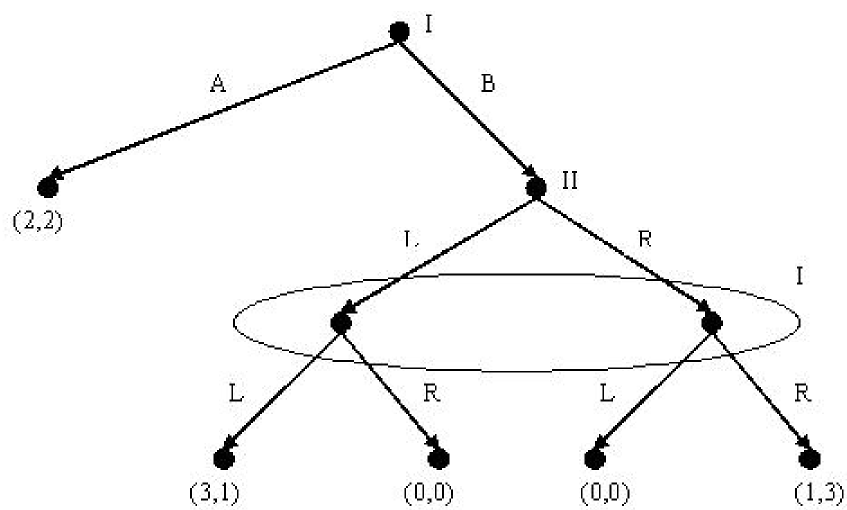

Figure 1:

Solution:

(a) Player I has two options, A or B. If he selects option A then both players get €2 and the game is over. If he selects option B then it is Player II turn to play. Player II may select one of two options, L or R. Now, not knowing the selection made by Player II, it is Player I turn to play. Player I may select one of two options, L or R. If his option differs from that of Player II then both players get €0. Otherwise, Player I gets €3 (and Player II gets €1) if he selects L, or Player I gets €1 (and Player II gets €3) if he selects R.
Let us find the equivalent strategic form of the game whose Kuhn tree is given in Figure 1. Player I has two information sets. He has 3 pure strategies denoted as follows:

(A) : choose A in his first move;
(B, L) : choose B in his first move and then choose L;
(B, R) : choose B in his first move and then choose R.

Therefore, $X = \{(A), (B, L), (B, R)\}$. Player II has only one information set. Hence, $Y = \{L, R\}$. The payoff matrix to both players is:

$$\begin{array}{c} \\ (A) \\ (B,L) \\ (B,R) \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{bmatrix} (2,2) & (2,2) \\ (3,1) & (0,0) \\ (0,0) & (1,3) \end{bmatrix} \end{array}.$$

(b) The payoff matrix to Player I is

$$\begin{array}{c} \\ (A) \\ (B,L) \\ (B,R) \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{bmatrix} 2 & 2 \\ 3 & 0 \\ 0 & 1 \end{bmatrix} \end{array}.$$

The element in the first row and second column is a saddle point (it is simultaneously a row min and a column max). Hence, the maxmin strategy for Player I is to play his pure strategy (A), in which case the outcome of the game will be $(2, 2)$.

(c) We put an asterisk after each Player I's payoffs that is maximum of its column and, we put an asterisk after each Player II's payoffs that is maximum of its row to get

$$\begin{array}{c} \\ (A) \\ (B,L) \\ (B,R) \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{bmatrix} (2,2^*) & (2^*,2^*) \\ (3^*,1^*) & (0,0) \\ (0,0) & (1,3^*) \end{bmatrix} \end{array}$$

and we observe that $((B, L), L)$ and (A, R) are the two pure Nash equilibria.

(d) If Player I does not play (A) then the game becomes the following

$$\begin{array}{c} \\ (B,L) \\ (B,R) \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{bmatrix} (3^*,1^*) & (0,0) \\ (0,0) & (1^*,3^*) \end{bmatrix} \end{array}.$$

Thus, there is no natural outcome of this game. There are two pure Nash equilibria $((B, L), L)$ or $((B, R), L)$ but one player prefers one and the other player prefers the other.

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2. Army A has a single plane with which it can attack one of three possible targets $1, 2,$ and 3. Army D has one anti-aircraft gun that can be assigned to defend one of these three targets. Army A can destroy a target only if the target is undefended and A attacks it. The value of destroying target k is k for army A. Similarly, the value of defending an attack on target k is k for army D. Both armies take their actions simultaneously, so the situation is summarized by the following normal form game:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ \begin{bmatrix} (0,1) & (1,0) & (1,0) \\ (2,0) & (0,2) & (2,0) \\ (3,0) & (3,0) & (0,3) \end{bmatrix} \end{array}.$$

Find as many strategic equilibria as you can, including the mixed one (Suggestion: eliminate one of the pure strategies of each player by domination).

Solution: First, note that pure strategy A_1 is strictly dominated by $\frac{3}{5}A_2 + \frac{2}{5}A_3$ and therefore cannot be a part of a Nash equilibrium. Once A_1 is eliminated, D_1 is dominated by any mixture of D_2 and D_3, so that the game reduces to the following 2×2 bimatrix game,

$$\begin{array}{cc} & \begin{array}{cc} D_2 & D_3 \end{array} \\ \begin{array}{c} A_2 \\ A_3 \end{array} & \left[\begin{array}{cc} (0,2) & (2,0) \\ (3,0) & (0,3) \end{array}\right] \end{array}, \tag{1}$$

for which there is no PSE. The game is defined by the pair of matrices

$$A = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Let (p_1, q_1) be a pair of equalizing strategies for B, which are defined by

$$p_1 = \left(\frac{3}{5}, \frac{2}{5}\right), \qquad q_1 = \left(\frac{3}{5}, \frac{2}{5}\right).$$

Denoting by $g_2(p, q)$ the average payoff to Player II if Player I uses the mixed strategy p and Player II uses the mixed strategy q then, from the definition of equalizing strategy,

$$g_2(p_1, q_1) = g_2(p_1, q), \qquad \text{for every mixed strategy } q.$$

Now, let (p_2, q_2) be a pair of equalizing strategies for

$$A^T = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix},$$

which are defined by

$$p_2 = \left(\frac{2}{5}, \frac{3}{5}\right), \qquad q_2 = \left(\frac{3}{5}, \frac{2}{5}\right).$$

Denoting by $g_1(p, q)$ the average payoff to Player I if Player I uses the mixed strategy p and Player II uses the mixed strategy q then, from the definition of equalizing strategy,

$$g_1(q_2, p_2) = g_1(q, p_2), \qquad \text{for every mixed strategy } q.$$

Hence, (p_1, p_2) is a Nash equilibrium. In other words, the only Nash equilibrium is to play

$$\left(\frac{3}{5}A_2 + \frac{2}{5}A_3; \frac{2}{5}D_2 + \frac{3}{5}D_3\right).$$

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3. Consider the cooperative TU bimatrix game:

$$\begin{bmatrix} (3,2) & (4,1) & (4,2) \\ (4,2) & (2,3) & (4,1) \\ (1,3) & (3,0) & (4,3) \end{bmatrix}.$$

(a) Find the TU-values.

(b) Find the associated side payment.

(c) Find the optimal threat strategies.

Solution:

(a) The maximum total payoff is $\sigma = 7$, with payoff (4, 3). The difference matrix is

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ -2 & 3 & 1 \end{bmatrix}.$$

The last row and last column are dominated, so

$$\delta = \mathrm{Val}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = 7/5.$$

Therefore, the TU solution is

$$\varphi = \left(\frac{\sigma+\delta}{2}, \frac{\sigma-\delta}{2}\right) = \left(\frac{21}{5}, \frac{14}{5}\right).$$

(b) Once the game is played and the outcome is $(4,3)$, Player II should pay $1/5$ to Player I.

(c) The threat strategies are $p = (3/5, 2/5, 0)$ and $q = (4/5, 1/5, 0)$.

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4. (a) Define what it means for a vector $(\bar{u}, \bar{v}) \in S$, where S is the NTU-feasible set, to be Pareto optimal in a two-player NTU game.

   (b) Consider the cooperative NTU bimatrix game:

$$\begin{bmatrix} (2,4) & (6,0) \\ (9,1) & (3,4) \end{bmatrix}.$$

   Let $(u^*, v^*) = (1, 0)$ be the disagreement point (or threat point, or status-quo point). Find the NTU-value (*i.e.*, the Nash bargaining solution).

Solution:

(a) A vector $(\bar{u}, \bar{v}) \in S$ is Pareto optimal if the only point $(u, v) \in S$ such that $u \geq \bar{u}$ and $v \geq \bar{v}$ is $(u, v) = (\bar{u}, \bar{v})$ itself.

(b) The Pareto optimal boundary is the line segment from $(3,4)$ to $(9,1)$. The equation of this line is $v - 4 = (-1/2)(u-3)$ or $v = (11-u)/2$. We seek the point on this line that minimizes $(u-1)v = (u-1)(11-u)/2$. Setting the derivative to zero gives $u = 6$, which gives $v = 5/2$. Thus $(6, 5/2)$ is the NTU solution since it is on the line segment.

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5. Consider the three-person game in coalitional form with characteristic function,

$$\begin{array}{llll} & \bar{v}(\{1\}) = 0 & \bar{v}(\{1,2\}) = 2 & \\ \bar{v}(\emptyset) = 0 & \bar{v}(\{2\}) = 1 & \bar{v}(\{1,3\}) = 3 & \bar{v}(\{1,2,3\}) = 10 \\ & \bar{v}(\{3\}) = 2 & \bar{v}(\{2,3\}) = 6 & \end{array}$$

   (a) How would you find the least rational core? (establish the linear program)

   (b) How would you find the nucleolus? Be succinct.

Solution: The normalized characteristic function v is defined by

$$\begin{array}{llll} & v(\{1\}) = 0 & v(\{1,2\}) = 2/10 & \\ v(\emptyset) = 0 & v(\{2\}) = 1/10 & v(\{1,3\}) = 3/10 & v(\{1,2,3\}) = 1 \\ & v(\{3\}) = 2/10 & v(\{2,3\}) = 6/10 & \end{array}$$

The set of imputations is the set X defined by

$$X = \{x \colon x_1 + x_2 + x_3 = 1, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}.$$

The reasonable set is the set of x's in X such that

$$
\begin{aligned}
x_1 &\le \max\{v(\{1,2,3\})-v(\{2,3\}), v(\{1,2\})-v(\{2\}), v(\{1,3\})-v(\{3\}), v(\{1\})-v(\emptyset)\}\\
&\le \max\{4/10, 1/10, 1/10, 0\} = 4/10\\
x_2 &\le \max\{v(\{1,2,3\})-v(\{1,3\}), v(\{1,2\})-v(\{1\}), v(\{2,3\})-v(\{3\}), v(\{2\})-v(\emptyset)\}\\
&\le \max\{7/10, 2/10, 4/10, 1/10\} = 7/10\\
x_3 &\le \max\{v(\{1,2,3\})-v(\{1,2\}), v(\{2,3\})-v(\{2\}), v(\{1,3\})-v(\{1\}), v(\{3\})-v(\emptyset)\}\\
&\le \max\{8/10, 3/10, 5/10, 2/10\} = 8/10.
\end{aligned}
$$

The least rational core, denoted $C^+(\epsilon_1)$ is the nonempty set of those x's in the reasonable set such that the excess $e(S,x)$, for each nontrivial subset S of $\{1,2,3\}$, is less than or equal to ϵ_1 and such that ϵ_1 is minimum. The value of ϵ_1 is the optimal value of the following LP

$$\min_{\epsilon,x}\{\epsilon\colon x\in X, e(S,x)\le\epsilon,\ \text{for all nonempty } S\subset\{1,2,3\}\}. \tag{2}$$

where

$$
\begin{aligned}
e(\{1\},x) &\equiv & -x_1 & \quad\le\epsilon & (3)\\
e(\{2\},x) &\equiv & 1/10-x_2 & \quad\le\epsilon & (4)\\
e(\{3\},x) &\equiv & 2/10-x_3 & \quad\le\epsilon & (5)\\
e(\{1,2\},x) &\equiv & 2/10-x_1-x_2\equiv x_3-8/10 & \quad\le\epsilon & (6)\\
e(\{1,3\},x) &\equiv & 3/10-x_1-x_3\equiv x_2-7/10 & \quad\le\epsilon & (7)\\
e(\{2,3\},x) &\equiv & 6/10-x_2-x_3\equiv x_1-4/10 & \quad\le\epsilon. & (8)
\end{aligned}
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6. Let X^1 be the optimal solution set of (2) and let ϵ_1 be the optimal value. If X^1 is singleton then its element is the nucleolus, otherwise, we must reiterate in the following way. We remove from (3)-(8) those inequalities that are satisfied as equalities at every $x\in X^1$ and impose them as equalities to ϵ_1. Then, we reiterate for $k=2,3,\ldots$ until we find X^k singleton. Then, its element is the nucleolus. ∎