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ON THE MINIMUM RANK AMONG POSITIVE SEMIDEFINITE MATRICES WITH A GIVEN GRAPH*

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Abstract. Let $\mathcal{P}(G)$ be the set of all positive semidefinite matrices whose graph is G, and $\mathrm{msr}(G)$ be the minimum rank of all matrices in $\mathcal{P}(G)$. Upper and lower bounds for $\mathrm{msr}(G)$ are given and used to determine $\mathrm{msr}(G)$ for some well-known graphs, including chordal graphs, and for all simple graphs on less than seven vertices.

Key words. rank, positive semidefinite, graph of a matrix

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1. Introduction.

If A is an n-by-n Hermitian matrix, then its graph $G(A)$ is the undirected, simple graph on vertices $\{1, 2, \ldots, n\}$, which has an edge between vertices i and j if and only if the i, j entry of A is nonzero and $i \neq j$. The graph is independent of the real diagonal entries of A. The set of all Hermitian matrices that share a common graph G is denoted $\mathcal{H}(G)$: $\mathcal{H}(G) = \{ A \mid A = A^*, G(A) = G \}$. If G is a simple connected graph, then matrices in $\mathcal{H}(G)$ may be viewed as the discrete version of the continuous Schrödinger operators with magnetic fields [3].

The possible multiplicities of the eigenvalues among matrices in $\mathcal{H}(G)$ have been of much recent interest [6, 7, 9, 10, 11, 13]. It is known, for example, that if G is a tree, then the smallest eigenvalue of any matrix in $\mathcal{H}(G)$ has multiplicity one [10, Corollary

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3.9]. This implies that any Hermitian positive semidefinite (psd) matrix whose graph is a tree has rank at least $n-1$. The Laplacian matrix of a tree on n vertices is a psd matrix with rank equal to $n-1$ [14]. A converse to this statement, that, for any nontree the minimum rank of a psd matrix is less than $n-1$, was proved independently (Lemma 5, [8] and Theorem 4.1, [17]). This raises the following interesting question, given a graph G, what is the minimum rank among psd matrices in $\mathcal{H}(G)$?

Let $\mathcal{P}(G)$ denote the psd matrices in $\mathcal{H}(G)$. Define the *minimum semidefinite rank of* G, $\operatorname{msr}(G)$, as $\min\{\operatorname{rank} A : A \in \mathcal{P}(G)\}$. We present here some results about $\operatorname{msr}(G)$, which give $\operatorname{msr}(G)$ for every chordal graph and for most graphs on fewer than seven vertices. The few exceptions can be handled by separate arguments. It is equally interesting to find the minimum psd rank over the symmetric real matrices instead of Hermitian matrices. It is not known if these two problems are different, though there can be differences in some related problems [12].

If G is not connected, it is clear that $\operatorname{msr}(G)$ is the sum of the minimum semidefinite ranks of each of G's connected components, so that we may (and do) confine our attention to connected graphs. Note that, if G is a connected graph with two or more vertices, the diagonal entries of $A \in \mathcal{P}(G)$ are positive.

2. Lower bounds using induced subgraphs.

We will obtain several lower bounds using induced subgraphs. An *induced subgraph* H of a graph G is obtained by deleting all vertices except for the vertices in a subset S. Since a principal submatrix of a psd matrix is psd [5, p. 397], and the rank of a submatrix can never be greater than that of the matrix, we have the following.

LEMMA 2.1. *If H is an induced subgraph of a connected graph G, then* $\operatorname{msr}(H) \leq \operatorname{msr}(G)$.

Equality can occur in the inequality of Lemma 2.1 in important ways; of course, strict inequality is common. One case of equality is that in which the induced subgraph is the result of the deletion of a duplicate vertex from G. For a vertex w, let $n(w)$ denote the set of all vertices adjacent to w. The *closed neighborhood* of w is $n(w) \cup \{w\}$. A vertex u is a *duplicate* of a vertex v of G if u and v are adjacent, and their closed neighborhoods are the same. We denote the induced subgraph of G resulting from the deletion of a vertex u by $G-u$. We then have the following.

PROPOSITION 2.2. *Let G be a connected graph on three or more vertices. If u is a duplicate vertex of v in G, then* $\operatorname{msr}(G-u) = \operatorname{msr}(G)$.

Proof. From Lemma 2.1, $\operatorname{msr}(G-u) \leq \operatorname{msr}(G)$. Let $A' \in \mathcal{P}(G-u)$ be a psd matrix such that $\operatorname{rank} A' = \operatorname{msr}(G-u)$. By permuting the rows and columns of A' let the first row and column of A' correspond to the vertex v. If $A' = B^*B$, then consider $A = \begin{bmatrix} B^* \\ e_1^{\mathrm{T}} B^* \end{bmatrix} \begin{bmatrix} B & Be_1 \end{bmatrix}$ where $e_1^{\mathrm{T}} = (1, 0, \dots, 0)$. Then $\operatorname{rank} A = \operatorname{rank} A'$ and $A \in \mathcal{P}(G)$. Thus $\operatorname{msr}(G) \leq \operatorname{msr}(G-u)$. ∎

From a sequential deletion of duplicate vertices and application of Proposition 2.2 we get the following.

COROLLARY 2.3. *If H is the induced subgraph of a connected graph G obtained by a sequential deletion of duplicate vertices of G and H has at least two vertices, then* $\operatorname{msr}(H) = \operatorname{msr}(G)$.

Remark 2.4. As an easy consequence of Corollary 2.3, we obtain that, for $n \geq 2$, $\operatorname{msr}(K_n) = 1$ where K_n denotes the complete graph on n vertices. Note that Proposition 2.2 is incorrect if applied to two *nonadjacent* vertices with the same neighbors. To see this, let G be K_4 minus an edge. Deletion of a degree 3 vertex gives $\operatorname{msr}(G) = 2$ using Proposition 2.2, but deletion of a degree 2 vertex results in K_3 whose msr equals one.

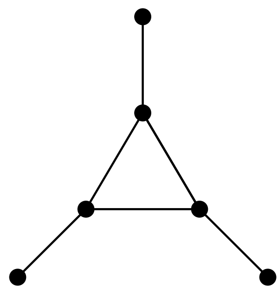

FIG. 2.1. $\mathrm{fm}(G) = 4$.

Another important application of Lemma 2.1 is that in which H is an induced tree on the maximum possible number of vertices as we know the msr for any tree. For a graph G, we consider its "tree size," denoted $\mathrm{ts}(G)$, which is the number of vertices in a maximum induced tree [4]. As already noted, when T is a tree, $\mathrm{msr}(T)$ is one less than the number of vertices of T. This fact, combined with Lemma 2.1, immediately gives the following.

LEMMA 2.5. *If G is a connected graph,* $\mathrm{msr}(G) \geq \mathrm{ts}(G) - 1$.

As mentioned in the introduction, equality in Lemma 2.5 occurs whenever G is a tree. It also occurs for any nontree G on n vertices for which $\mathrm{ts}(G) = n - 1$; in this case $\mathrm{msr}(G) \geq n - 2$ by Lemma 2.5, and $\mathrm{msr}(G) \leq n - 2$ because G is not a tree. Thus $\mathrm{msr}(G) = n - 2$. For example, if G is a cycle on n vertices, the tree size is $n - 1$ (because deletion of any one vertex leaves a path on $n - 1$ vertices). Therefore, the msr of a cycle on n vertices is $n - 2$ (cf. [17, Theorem 4.3]).

For an induced forest of G with components $T_1, T_2, \ldots, T_k$, count $\mathrm{ts}(T_1) + \mathrm{ts}(T_2) + \cdots + \mathrm{ts}(T_k) -$ (the number of components that are not isolated vertices). Among all the induced forests of G maximize this count and call this result $\mathrm{fm}(G)$, the "forest measure" of G. Any isolated vertices occurring in an induced subgraph of a connected graph G contribute 1, rather than 0, to $\mathrm{msr}(G)$, as an irreducible psd matrix has positive diagonal entries. We then have the following.

PROPOSITION 2.6. *If G is a connected graph, then* $\mathrm{msr}(G) \geq \mathrm{fm}(G) \geq \mathrm{ts}(G) - 1$.

Figure 2.1 illustrates that strict inequality is possible in the second inequality of Proposition 2.6, as $\mathrm{fm}(G) = 4$ by deleting any single interior vertex.

One special case of an induced forest is an induced set of isolated vertices. The maximum cardinality of such a set is the *independence number* $i(G)$, the greatest number of vertices among which there are no edges. Clearly $\mathrm{fm}(G) \geq i(G)$, so that we have the following.

COROLLARY 2.7. *For a connected graph G,* $\mathrm{msr}(G) \geq i(G)$.

Suppose G is a connected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. We call a set of vectors $\overrightarrow{V} = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}\}$ in $\mathbb{C}^m$ a *vector representation* (or *orthogonal representation*) of G if

$$\begin{bmatrix} \overrightarrow{v_1} \\ \overrightarrow{v_2} \\ \vdots \\ \overrightarrow{v_n} \end{bmatrix} \begin{bmatrix} \overrightarrow{v_1} \\ \overrightarrow{v_2} \\ \vdots \\ \overrightarrow{v_n} \end{bmatrix}^* = A \in \mathcal{P}(G).$$

In other words, we associate a vector $\overrightarrow{v_i} \in \mathbb{C}^m$ to each vertex $v_i \in V(G)$ such that, for $i \neq j$, $\langle \overrightarrow{v_i}, \overrightarrow{v_j} \rangle \neq 0$ if v_i and v_j are adjacent in G, and $\langle \overrightarrow{v_i}, \overrightarrow{v_j} \rangle = 0$ if v_i and v_j

are not adjacent. Since every psd matrix $A \in \mathcal{P}(G)$ can be written as $A = B^*B$ for some matrix B, we can always find a vector representation of $G(A)$ that produces A. Also, the rank of the matrix and the dimension of the span of the vectors in the vector representation (which we call the *rank* of the vector representation) are always the same [5, p. 408].

We end this section by giving a sufficient condition on G so that $\operatorname{msr}(G) = \operatorname{ts}(G) - 1$. To prove the result we need the following lemma.

LEMMA 2.8. *Suppose $X_1, \ldots, X_m$, $X_i \subseteq \mathbb{C}^n$ for $1 \le i \le m$, are vector representations of subgraphs $G_1, \ldots, G_m$ of a connected graph G such that*

- *for every pair of adjacent vertices v, w of G, there exists an i such that v and w are adjacent in G_i; and*
- *for every pair of vertices v, w of G that are not adjacent, if $\overrightarrow{x_v}$ represents vertex v in X_i and $\overrightarrow{x_w}$ represents vertex w in X_j, $\langle \overrightarrow{x_v}, \overrightarrow{x_w} \rangle = 0$.*

Then there exists a vector representation X of G, with

$$\operatorname{rank} X \le \operatorname{rank}\left(\bigcup_{1\le i\le m} \operatorname{span} X_i\right) \le \sum_{1\le i\le m} \operatorname{rank} X_i.$$

Proof. We prove the statement for the case of two vector representations as the result can be easily generalized. Let $X_1 = \{\overrightarrow{x_i}\}$ and $X_2 = \{\overrightarrow{w_i}\}$ be vector representations of subgraphs G_1 and G_2 of a graph G. Extend X_1 and X_2 to represent all of the vertices of G by adding copies of the zero vector if need be. We claim there exists $c \in \mathbb{R}$ such that $\{\overrightarrow{x_i} + c\overrightarrow{w_i}\}$ is a vector representation of G.

If $(v_i, v_j) \notin E$, then $\langle \overrightarrow{x_i}, \overrightarrow{x_j} \rangle = \langle \overrightarrow{w_i}, \overrightarrow{w_j} \rangle = \langle \overrightarrow{x_i}, \overrightarrow{w_j} \rangle = \langle \overrightarrow{w_i}, \overrightarrow{x_j} \rangle = 0$. This implies that $\langle \overrightarrow{x_i} + c\overrightarrow{w_i}, \overrightarrow{x_j} + c\overrightarrow{w_j} \rangle = 0$ for any $c \in \mathbb{C}$. If v_i and v_j are adjacent, then $\{\langle \overrightarrow{x_i} + c\overrightarrow{w_i}, \overrightarrow{x_j} + c\overrightarrow{w_j} \rangle\}$ is a set of quadratics in c having finitely many roots. Thus we may choose $c \in \mathbb{R}$ so that $\{\overrightarrow{x_i} + c\overrightarrow{w_i}\}$ is a vector representation of G. ◻

Suppose T is a maximum induced tree. If w is a vertex not belonging to T, denote by $\mathcal{E}(w)$ the edge set of all paths in T between every pair of the vertices of T that are adjacent to w.

THEOREM 2.9. *For a connected graph G,* $\operatorname{msr}(G) = \operatorname{ts}(G) - 1$ *if the following condition ⊛ holds:*

⊛ There exists a maximum induced tree T such that, for u and w not on T, $\mathcal{E}(u) \cap \mathcal{E}(w) \ne \varnothing$ if and only if u and w are adjacent in G.

Proof. If G is a tree, we have already seen that $\operatorname{msr}(G) = \operatorname{ts}(G) - 1$. If G is not a tree, we will cover G with subgraphs that have vector representations satisfying the conditions of Lemma 2.8. If ⊛ holds for a maximum induced tree T of G, then every vertex w not on T must be adjacent to some vertex on T. Moreover, by the definition of T, w is adjacent to at least two vertices of T. Assign an orthonormal set of vectors $\{\overrightarrow{x_e}\}$ of dimension $(\operatorname{ts}(G) - 1)$ to the edges of T, one vector per edge. If $v \in V(T)$, assign the vector $\overrightarrow{v} = \sum_e \overrightarrow{x_e}$ to v, where the summation is over all edges incident to v. This gives a vector representation $\overrightarrow{T}$ of T.

For any path $p = (e_1, e_2, \ldots, e_m)$ in T, let

$$\overrightarrow{p} = \sum_{j=1}^{m} (-1)^j \overrightarrow{x_{e_j}}.$$

Given a vertex w not on T and an adjacent vertex v_1 on T, w must have another neighbor v_2 on T. If p is a path between v_1 and v_2 in T, letting $\overrightarrow{p}$ represent w and $\overrightarrow{T}$ represent T yields a vector representation of a subgraph of G containing the edge between w and v_1.

Given two vertices w_1 and w_2 not on T that are adjacent, by $\circledast$ there exist intersecting paths p_1 and p_2 in T so that the end vertices of p_i are neighbors of w_i, $i = 1, 2$. Letting $\overrightarrow{p_i}$ represent w_i for $i = 1, 2$ and $\overrightarrow{T}$ represent T yields a representation of a subgraph of G containing the edge connecting w_1 and w_2.

By construction, these representations cover all the edges of G and are contained in span$\{\overrightarrow{x_e}\colon e$ an edge of $T\}$. We now show that these representations satisfy the conditions of Lemma 2.8. If v and w are adjacent in G, we have explicitly constructed above a representation of a subgraph of G in which v and w are adjacent.

If v and w are not adjacent, there are three cases to consider. First, if v and w are both vertices in T, then in any two representations, v and w are represented by the corresponding vectors in $\overrightarrow{T}$, which are orthogonal. For other cases, first notice that, if a vertex w is not on T, then w is represented by $\overrightarrow{p}$ derived from a path p. If v is a vertex on T not adjacent to w, then v cannot be an endpoint of p. Thus the vector representing v in $\overrightarrow{T}$ is orthogonal to $\overrightarrow{p}$. Suppose v and w are both not on T and are not adjacent in G. The vectors $\overrightarrow{q}$ and $\overrightarrow{p}$ representing v and w, respectively, are derived from paths q and p, respectively. By $\circledast$ the paths p and q have no edges in common and thus $\overrightarrow{p}$ and $\overrightarrow{q}$ must be orthogonal. Applying Lemma 2.8 we get $\mathrm{msr}(G) \leq \mathrm{ts}(G) - 1$. ☐

3. Chordal graphs. The sum of two psd matrices is psd, and the rank of a sum is never more than the sum of the ranks [5, p. 13]. If we cover all of the edges of a graph G with (not necessarily induced) subgraphs of known msr, this can lead to useful upper bounds for $\mathrm{msr}(G)$. First, suppose that G is labeled and that $G_1, \ldots, G_k$ are (labeled) subgraphs of G, that is, each $G_i, i = 1, \ldots, k$ is the result of deleting some edges and/or vertices from G. We say that $G_1, \ldots, G_k$ *cover* G if each vertex of G is a vertex of at least one G_i, and for every pair of adjacent vertices v, w of G, v and w are adjacent in at least one G_i. The cover $C_1, \ldots, C_k$ of G is called a *clique cover* of G if each of $C_1, \ldots, C_k$ is a clique of G. The *clique cover number* $\mathrm{cc}(G)$ (see [15]) of G is the minimum value of k for which there is a clique cover $C_1, \ldots, C_k$ of G.

PROPOSITION 3.1. *For any simple connected graph* G*,* $\mathrm{msr}(G) \leq \mathrm{cc}(G)$*.*

Proof. The proof follows from Lemma 2.8 and Remark 2.4. ☐

Since the clique cover number of a cycle on $n \geq 4$ vertices is n but its msr is $n-2$, strict inequality is possible in Proposition 3.1.

Given a vector representation $\overrightarrow{V}$ of G, with $\overrightarrow{v}$ representing vertex v, replace each vector $\overrightarrow{w} \in \overrightarrow{V}$ with the orthogonal projection

$$\overrightarrow{w} - \frac{\langle \overrightarrow{v}, \overrightarrow{w} \rangle}{\langle \overrightarrow{v}, \overrightarrow{v} \rangle} \overrightarrow{v}$$

to yield a set of vectors denoted $\overrightarrow{V} \ominus \overrightarrow{v}$. It is easily verified that $\mathrm{rank}(\overrightarrow{V})$ is one more than $\mathrm{rank}(\overrightarrow{V} \ominus \overrightarrow{v})$.

Consider the graph corresponding to $\overrightarrow{V} \ominus \overrightarrow{v}$. It is obtained from the original graph G, first by removing the vertex v and then modifying the graph in the following manner: For $u, w \in n(v)$, if (u, w) is not an edge of G, then (u, w) is an edge of the modified graph and if (u, w) is an edge of G, then (u, w) may or may not be an edge of the modified graph. Notice in the latter case that the "may or may not" depends on the choice of vector representation $\overrightarrow{V}$. In what follows, we consider graphs which have multiple edges. This allows us to define below a graph $G \ominus v$, which better captures the relationship between $\overrightarrow{V} \ominus \overrightarrow{v}$ and the "orthogonal removal of vertex v."

Following van der Holst [17], let G be an undirected graph with no loops but possibly multiple edges, with vertex set $V = \{1, 2, \ldots, n\}$. Let $\mathcal{H}_G$ be the set of all n-by-n Hermitian matrices $A = [a_{ij}]$ such that

- $a_{ij} \neq 0$ if i and j are connected by exactly one edge;
- $a_{ij} = 0$ if i and j are not adjacent, and $i \neq j$.

Notice that we make no restriction on a_{ij} if i and j are connected by more than one edge. Now, $\overrightarrow{V} = \{\overrightarrow{v_1}, \ldots, \overrightarrow{v_n}\}$ in $\mathbb{C}^m$ is a *vector representation* of a graph G with multiple edges when $\langle \overrightarrow{v_i}, \overrightarrow{v_j} \rangle \neq 0$ if i and j are connected by a single edge and $\langle \overrightarrow{v_i}, \overrightarrow{v_j} \rangle = 0$ if i and j are not adjacent.

Let G be a graph (with multiple edges). The graph $G \ominus v$, called the *orthogonal removal of v from G*, is obtained as follows: In the induced subgraph $G - v$ of G, between any $u, w \in n(v)$ add $e - 1$ edges, where e is the sum of the number of edges between u and v and the number of edges between w and v.

Remark 3.2. If $\overrightarrow{V}$ is a vector representation of a graph G, then $\overrightarrow{V} \ominus \overrightarrow{v}$ is a vector representation of $G \ominus v$. As mentioned earlier, this process results in a representation that has rank one less than rank $\overrightarrow{V}$. Unfortunately, $\operatorname{msr}(G) - \operatorname{msr}(G \ominus v)$ may be arbitrarily large as demonstrated by the complete bipartite graph $K_{2,n}$: For $n \geq 3$, by Corollary 2.7, $\operatorname{msr}(K_{2,n}) \geq n$, but the orthogonal removal of a vertex from the smaller independent set yields the complete graph on $n+1$ vertices, K_{n+1}, and $\operatorname{msr}(K_{n+1}) = 1$ by Remark 2.4.

We say that subgraphs $G_1, \ldots, G_m$ *cover* a graph G with multiple edges if each vertex of G is a vertex of at least one G_i and, for every pair of vertices v and w of G joined by *exactly* one edge, there exists an i such that v and w are joined by *exactly* one edge in G_i. We now restate Lemma 2.8 for graphs with multiple edges.

LEMMA 3.3. *Suppose $X_1, \ldots, X_m$, $X_i \subseteq \mathbb{C}^n$ for $1 \leq i \leq m$, are vector representations of subgraphs $G_1, \ldots, G_m$ of a connected graph G (with multiple edges) such that*

- *$G_1, \ldots, G_m$ cover G;*
- *for every pair of vertices v, w that are not adjacent in G, if $\overrightarrow{x_v}$ represents vertex v in X_i and $\overrightarrow{x_w}$ represents vertex w in X_j, $\langle \overrightarrow{x_v}, \overrightarrow{x_w} \rangle = 0$.*

Then there exists a vector representation X of G, with

$$\operatorname{rank} X \leq \operatorname{rank} \left(\bigcup_{1 \leq i \leq m} \operatorname{span} X_i \right) \leq \sum_{1 \leq i \leq m} \operatorname{rank} X_i.$$

Recall that a vertex v such that $n(v)$ induces a complete graph is said to be *simplicial*.

LEMMA 3.4. *Suppose v is a simplicial vertex of a connected graph G that is joined to at least one neighbor by exactly one edge. Then* $\operatorname{msr}(G) = \operatorname{msr}(G \ominus v) + 1$.

Proof. From Remark 3.2, we have that $\operatorname{msr}(G) \geq \operatorname{msr}(G \ominus v) + 1$. From Remark 2.4, we may find a vector representation of rank one of the subgraph of G induced by v and its neighbors. Choosing this representation to be orthogonal to a representation of $G \ominus v$, we may apply Lemma 3.3 to see that $\operatorname{msr}(G) \leq \operatorname{msr}(G \ominus v) + 1$. ∎

The following corollary simplifies finding the minimum rank of graphs with *pendant* vertices, which are simply vertices of degree 1. This corollary is also found in [17, Lemma 3.6] with a different proof.

COROLLARY 3.5. *If a simple connected graph G has a pendant vertex v, then* $\mathrm{msr}(G) = \mathrm{msr}(G - v) + 1$.

A graph is said to be *chordal* if it has no induced cycles C_n with $n \geq 4$. It is known that every nonempty chordal graph has at least one simplicial vertex [2, p. 175]. A clique cover of a graph G with multiple edges is a collection of cliques of G that cover every *single* edge between the vertices of G. As before, the clique cover number of G $\mathrm{cc}(G)$ is the minimum number of cliques in a clique cover of G. We are now able to show that, for chordal graphs, the msr is the clique cover number.

THEOREM 3.6. *Let G be a connected chordal graph. Then* $\mathrm{msr}(G) = \mathrm{cc}(G)$.

Proof. Induct on the number of vertices of G. We start the induction with an edge. For graphs with three or more vertices, identify a simplical vertex v of G. If, in addition, v is a duplicate vertex, then $\mathrm{cc}(G - v) = \mathrm{cc}(G)$ and $\mathrm{msr}(G - v) = \mathrm{msr}(G)$. If v is not a duplicate vertex and not connected to any other vertex by exactly one edge, then $\mathrm{cc}(G - v) = \mathrm{cc}(G)$ and $\mathrm{msr}(G - v) = \mathrm{msr}(G)$.

Finally, if v is not a duplicate vertex and is connected to at least one other vertex by exactly one edge, we observe that $\mathrm{cc}(G \ominus v) = \mathrm{cc}(G) - 1$. To see this, when v is simplicial, there are multiple edges between each pair of vertices in $n(v)$ in $G \ominus v$. Thus remove exactly one clique from a minimum clique cover of G to obtain a clique cover of $G \ominus v$. Now using Lemma 3.4 we get $\mathrm{msr}(G) = \mathrm{cc}(G)$. ∎

4. Minimum psd rank for graphs on less than seven vertices. For all the graphs G with $|V(G)| \leq 6$, with a few exceptions listed below, we can determine $\mathrm{msr}(G)$ using results discussed in this paper. A catalog of these graphs can be found in [16]. Table 4.1 lists the minimum psd ranks of 142 *connected* graphs on 2 or more vertices but less than seven vertices using the numbering found in [16].

We now detail how to use the results of this paper to find the msr of the graphs listed in Table 4.1. The graphs G174, G175, G198, and G204 are the exceptional cases which cannot be handled by the results presented above. We provide alternate methods for these graphs.

The complete graphs G3, G7, G18, G52, and G208 have msr equal to 1 by Remark 2.4. As mentioned in the introduction, the msr of a tree is one less than the number of vertices. This gives the msr for the trees G3, G6, G13, G14, G29–31, G77–81, and G83.

Among the nontree, noncomplete graphs, the following 64 graphs are chordal: G15, G17, G34–36, G40–42, G45–47, G49, G51, G92–95, G97, G100, G102, G111–

TABLE 4.1

$\mathrm{msr}(G)$	Graph
5	G77–81 and G83.
4	G29–31, G92–100, G102–105, G111-115, G118, G120–125, G127–129, G135–139, G145–149, G152, G161, G162, G164, and G167.
3	G13, G14, G34-38, G40, G41, G43, G44, G46, G47, G117, G119, G126, G130, G133, G134, G140–144, G150, G151, G153, G154, G156–160, G163, G166, G168–175, G177–189, G192, G193, G196–198, G201, and G202.
2	G6, G15–17, G42, G45, G48-51, G165, G190, G191, G194, G195, G199, G200, and G203–207.
1	G3, G7, G18, G52, and G208.

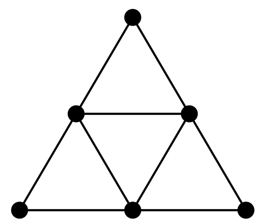

FIG. 4.1. $G163$.

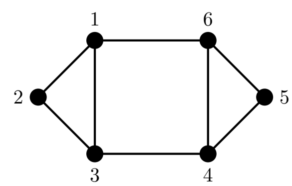

FIG. 4.2. $G152$.

115, G117, G119, G120, G123, G130, G133–139, G142, G144, G150, G156, G157, G160–165, G167, G177–181, G183, G191–193, G195, G200–G202, G205, and G207. Theorem 3.6 gives that the msr of a chordal graph is its clique cover number. For example, we have cc(G163) = msr(G163) = 3 (Figure 4.1).

There are 20 nonchordal graphs whose msr is 4. All but graph G152 (Figure 4.2) satisfy $\mathrm{ts}(G) = 5$. The discussion following Lemma 2.5 shows that, for these graphs, $\mathrm{msr}(G) = 4$. For G152, if we orthogonally remove simplicial vertices 2 and 5 and apply Lemma 3.4, we observe that msr(G152) = 4. In addition, G152 is not chordal, but $\mathrm{msr}(G) = \mathrm{cc}(G) = 4$, indicating that the converse to Theorem 3.6 is false.

Among the 32 nonchordal graphs whose msr is 3, G37, G38, G43, and G44 have $\mathrm{ts}(G) = 4$, hence they have $\mathrm{msr}(G) = 3$. The msr of G140, G141, G143, G158, and G159 is 3 by Corollary 3.5. A duplicate vertex is removed in G126, G153, G168, G169, G170, G172, G185, and G189, and the resulting graph on 5 vertices has msr equal to 3. The graphs G151, G154, G166, G171, G173, G182, G184, G186–G188, G196, and G197 satisfy the sufficient condition of Theorem 2.9. The exceptional cases are G174, G175, and G198. These three graphs could be handled using a construction as shown below or by applying Theorem 3.1 and Proposition 3.2 of [17] along with Lemma 2.5.

A maximum induced tree of G198 (Figure 4.3) is induced by $\{v_1, v_2, v_3, v_4\}$. Using the Laplacian matrix of this tree in the top left 4-by-4 block, we construct rows 5 and 6 to represent the graph G198,

$$\begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 1 \\ -1 & 2 & -1 & 0 & 1 & -1 \\ 0 & -1 & 2 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & 2 & 0 \\ 1 & -1 & 1 & -1 & 0 & 2 \end{bmatrix}.$$

The graph G198, with this rank 3 psd matrix, is an example which shows that the $\circledast$ condition of Theorem 2.9 is not necessary.

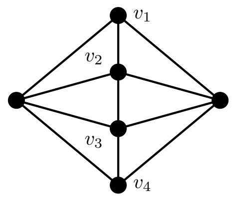

FIG. 4.3. $G198$.

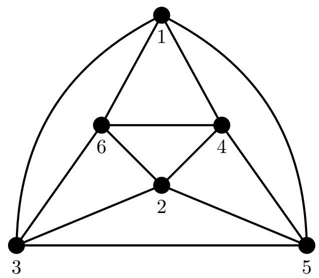

FIG. 4.4. $G204$.

Among the 9 nonchordal graphs whose msr is 2, G16 is a cycle on 4 vertices, while G50 and G203 satisfy the sufficient condition of Theorem 2.9. Removing one duplicate vertex from G48 and G206, and removing two duplicate vertices from G190, G194 and G199 reduce the graph to a known case. The one exceptional case is G204 (see Figure 4.4).

Suppose $\overrightarrow{e_1} = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\overrightarrow{e_2} = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Then we can write a vector representation for G204 as follows: $\overrightarrow{v_1} = \overrightarrow{e_1}$, $\overrightarrow{v_2} = \overrightarrow{e_2}$, $\overrightarrow{v_3} = 2\overrightarrow{e_1} + \overrightarrow{e_2}$, $\overrightarrow{v_4} = \overrightarrow{e_1} - 2\overrightarrow{e_2}$, $\overrightarrow{v_5} = \overrightarrow{e_1} + \overrightarrow{e_2}$, and $\overrightarrow{v_6} = \overrightarrow{e_1} - \overrightarrow{e_2}$. Thus msr(G204) = 2. Alternatively, we may use [1, Theorem 15].

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